Dozenal Division

of the

Circle

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Abstract

This article examines the division of the circle into one, two, three, and four dozen parts, assigning points to each division according to certain criteria. The division with the least number of points at the conclusion is considered the best. The result of this exploration shows that division of the circle into two dozen parts is the best in terms of practical application.

Dozenalists began by assuming that the circle would be divided into a dozen parts; however, largely through the influence and work of Tom Pendlebury, a division of the circle into two dozen parts, the semicircle (or straight line) being deemed the primary unit of angle, became mainstream among dozenalists. However, the recent popularity of 2π, often called τ, as a circle constant, supposedly as a replacement for π, among math enthusiasts has led to some renewed support for dividing the circle into a dozen parts, and the experience which led to abandoning this idea in earlier dozenal days has been largely forgotten. Consequently, this paper has recreated a number of experiments judging the utility of dividing the circle into one, two, three, and four dozen parts, with points being assigned to each.

The most advantageous division will be given one point; the next most two; and so forth. Ties will be broken by dividing the number of points evenly between the tied divisions. Consequently, the division with the least number of points at the end "wins," in the sense that it has been the most advantageous the most number of times.

Experiments will be broadly grouped here by section, with more specific experiments grouped by subsection and, where necessary, subsubsections.

The important thing to remember when reading this is that our primary concern is *not* mathematical purity, ease in complex derivations, or anything of that sort; our oncern is *ease of use* in the most common applications. This should help explain a variety of the judgments we have made.

1 Initial Concepts

The first thing to remember in this discussion is that there is really only one “natural” unit of angle: the radian. As many layman are not familiar with the concept of the radian, this probably bears some explanation.

1.1 The Radian

Consider a circle with a radius of exactly one unit. That unit may be a Grafut, an inch, a centimeter, whatever; it doesn’t matter, provided that we consider it as equal to “one.” We then set that circle on its edge and begin to roll it. Once we have rolled it a horizontal distance equal to its radius—equal to our “one unit”—it has rolled through a certain portion of its circumference. We then stop there, mark that part of the circumference that it’s rolled through, and lay it back down on its side to look at it.

We then draw the radius from the center of the circle to the point on the circumference of the circle that we started on, and we then draw another line from the center of the circle to the point on the circumference of the circle that we ended on. The angle between these two lines is called a *radian*.

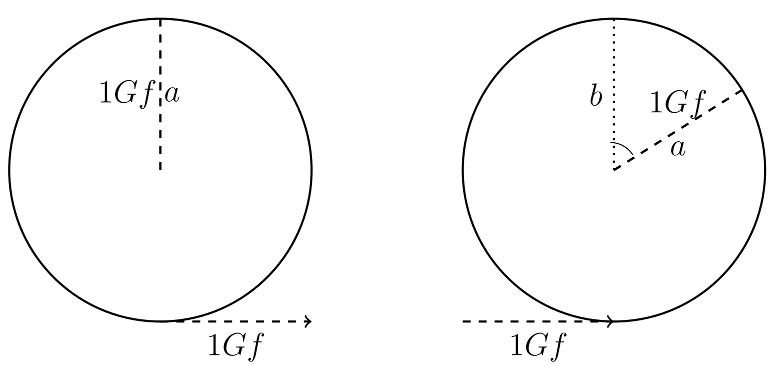

Figure 1: *Constructing the radian from a turning circle.*

This process is depicted graphically in Figure 1 on page 5. The first circle shows the radius, a, with a length of one Grafut. It also shows, on the bottom of its circumference, that the circle will be rolled to the right a distance of one Grafut. The second circle shows, at its bottom, an arrow indicating that it has been rolled a distance of one Grafut. We also see its radius, a, now changed in position due to the rotation of the circle. We draw a new radius, b, in the original position of radius a; the angle between these two is called a *radian*.

However, the radian is really even simpler than this. It's easy to describe it in terms of the circumference of a circle, but it's even easier to describe it in terms of a simple angle, equal to 1 radian. Figure 2 on page 6 depicts this process graphically.

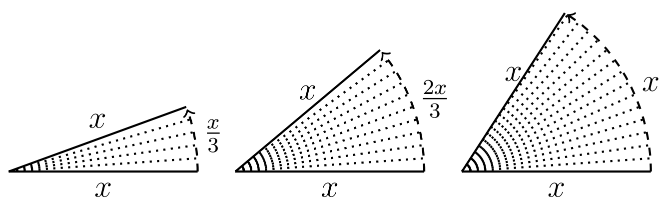

Figure 2: *Constructing the radian from an opening angle.*

Essentially, we start with one line, and we begin to form an angle. We do this by taking another line of the same length—it's important that it be the same length, in this case x—and begin to sweep it upwards. In the leftmost angle, we see that we've swept it upwards such that the arc swept out by the top line is equal to $\frac{x}{3}$ in length; in the second picture, we've doubled that, and it's equal to $\frac{2x}{3}$. In the third picture, though, we've swept out so much space that the length of the arc is *equal* to x. The angle that this arc subtends is one radian.

This angle is the same no matter what the length of the radius, and consequently is really the only natural unit of angle. As such, many important functions work best when angles are fed to them in radians; when using other units, such as degrees, we must apply some kind of conversion factor to get our answer back the way we need it.

Therefore, throughout our discussion of angle, we will not be asserting that any of our proposed divisions of the circle produce the most *natural* unit of angle; for that is simply the radian. We are instead investigating what produces the easiest unit of angle for practical use. This should be remembered throughout this work.

1.2 Symbols Used Throughout This Work

We will be using a number of symbols throughout this work fairly consistently; the most important of these are those which represent certain numbers of radians.

The number of radians in our proposed divisions of the circle are never even; for that reason, we will present them individually. We will also apply special Greek letters to each of these numbers, because they are *transcendental* numbers, ones with infinitely continuing digits showing no repeating pattern.

Most of the interesting numbers of radians also correspond to ratios of the circumference of a circle to its radius or diameter. Figure 3 on page 7 shows the four most important, which correspond to our four divisions of the circle.

Portion	Symbol	Value	Ratio
Whole	ς	6;34941696	$C = \varsigma r$
Half	π	3;18480949	$C = \pi d$
Third	ψ	2;11714632	$C = 3\psi r$
Quarter	η	1;6↊240484	$C = \eta 2d$

Figure 3: *Important numbers of radians and their ratios to the circumference of the circle and its radius.*

For Figure 3, d is the diameter of the circle and r is its radius. They are chosen according to which will make the ratio most clear. Of course, C is the circumference. The "value" column indicates what the symbol actually means as well as the number of radians in the given portion of the circle. That is, π radians is a half circle; ψ radians is the number of radians in a third of a circle; and so forth.

We have selected ς (a variant of sigma, pronounced "varsigma") for the number of radians in the full circle for good reason. Commonly τ is used for this; however, τ is an overloaded symbol that means too many different things already. ς bears a vague resemblance to c, which of course can be construed as standing for "circle," and it has no other mathematical meaning; therefore, it seems an ideal symbol.

We will also frequently be using c to represent the angle equal to a full circle, and u to represent the unit of angle, regardless of its size.

Other symbols will be used *ad hoc* and explained when appropriate.

2 Definitions of Angle

The first question to be explored here is what we're really measuring; are we measuring *rotations*, *turns*, or what? In other words, we must return to basics: what is an *angle*?

2.1 Geometric Definition—4, 1, 3, 2

The first definition of angle we'll explore is what we will term the *geometric definition*. This is the definition that we probably all learned first, in our very earliest geometry classes, long before being introduced to the Cartesian plane. At this stage we met *lines*—series of points extending infinitely in both directions, never intersecting with themselves; *line segments*, small portions of lines bounded on both ends by two points; and *rays*, parts of lines bounded on only one end by points.

Here we meet the quintessential definition of the angle, the most basic possible meaning for an angle, and the first meaning for an angle that is encountered by the young when learning mathematics. It is, therefore, an extremely important definition:

Definition 1. *An angle is the space between two rays sharing a common endpoint.*

The common endpoint shared by these two rays is called the *vertex*; an image of this most basic type of angle, the first encountered by learners of mathematics, is depicted in Figure 4 on page 9. This definition established, let us examine our four divisions of the circle and determine which best embodies it.

Two dozen This division of the cicle perfectly embodies this definition. It makes the *straight line* the fundamental unit of angle; a circle is simply the sum of the angles on either side of a single straight line. Because two

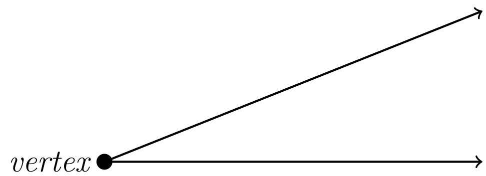

Figure 4: *The basic angle depicted as composed of two rays with a common endpoint.*

rays sharing a common endpoint can always be interpreted as an angle less than a straight line—a reflex angle can simply be reinterpreted as the explement of the actual angle—this division of the circle makes the largest possible angle the unit and all smaller angles fractions thereof. Consequently, this division wins this category, taking a single point.

Four dozen Dividing a circle into four dozen parts makes some sense according to this definition. This division of the circle makes the *right angle* its basic unit, and the right angle—two rays related perpendicularly to one another—is certainly an important special case which needs to be conveniently catered to. However, this division means that a great many angles will be greater than our unit of angle; this unnecessarily complicates calculations. Two points.

Three dozen There's not much to say for this division of the circle in most categories, and this one is no exception. However, it still takes third place due to the utter failure of the whole-circle definition; for this division of the circle at least enshrines an actual angle as the unit. Three points.

One dozen The division of the circle into one dozen parts misses this definition entirely. For the unit of angle, by this definition, is an angle of *zero*. Indeed, by this definition the unit of angle isn't an angle at all; there are not two rays sharing a common endpoint. It completely confuses the learner who has been taught this description of angles because the unit doesn't describe an angle at all, but rather simply a single ray. The division of the circle into one dozen parts utterly fails according to the most basic definition of an angle, a strong strike against a system which at first glance appears quite intuitive. Thus, this division completely fails as a unit of angle according to this definition. Four points.

Final verdict: two dozen; four dozen; three dozen; one dozen.

2.2 Directional Definition—3, 1, 4, 2

The *directional definition* of angles considers angles not merely in terms of common or intersecting points, but in terms of *changes in direction*. This is a very important concept of angle, because it corresponds with a concept we all deal with from very early on in our studies of mathematics: the *number line*.

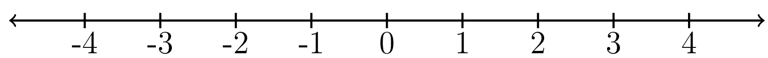

Figure 5: *A number line.*

As we can see here, the reversal of direction is considered quite simply as a negation of the first direction. We see a close parallel here with the division of the circle into two dozen parts, where turning in one direction is positive but turning in the other is negative (or greater than one reversal of direction, which amounts to the same thing).

This could also be considered the *semicircular definition*, since the angle of half a circle is a straight angle.

Definition 2. *An angle is a change in direction as measured from some original direction.*

It's clear from this definition that the maximum possible angle is a reversal of direction, since the more one turns after that the less real displacement occurs, until one returns to one's original orientation, when the change of direction is practically 0. Consequently, this section will focus on a full change of direction and less.

Figure 6 on page ↋ shows a full change of direction with the dozenal divisions thereof marked off; angle measurements are at each such division, in the order one dozen, two dozen, three dozen, and four dozen.

It's worth noting that these dozenal divisions of one full change of direction —a straight angle, or a turnabout—are equal to fifteen degrees, or $\frac{\pi}{10}$

(dozenal) radians; these multiples of angles are *extremely* common, probably the most common of all. They also correspond to hours of right ascension in terms of astronomy.

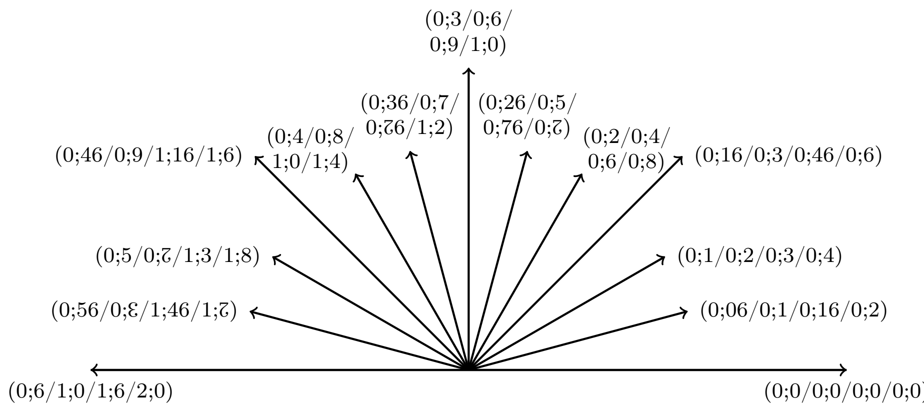

Figure 6: *An illustration of changes of direction in the various systems.*

It will come as no surprise here that the division into two dozen takes the prize for this definition of angle. It sets the maximum change in direction equal to one; this means that an extraordinarily large number of the most common divisions—really, all of them—are simple uncias.

The next most advantageous here is a close call between the division into one dozen and the division into four dozen. The one dozen division sets the maximum change of direction equal to one half, and reserves the whole for a turn so large that one is facing the same direction again. This may give it an advantage when considering angle under a different aspect,* but for now it simply confuses things. The division into four dozen makes half a full turn the unit; we wind up with 2;0 units in a full reversal. Since whole numbers are typically easier to deal with than fractions, and the four dozen gives us more whole numbers than the one dozen, the four dozen takes second place while the one dozen takes third. The three dozen, unsurprisingly, takes last.

Final verdict: two dozen, four dozen, one dozen, and three dozen.

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* *See supra*, Section 2.4, at 12.

2.3 Intersection Definition—3, 1, 4, 2

Once we've learned about geometric angles, we're next introduced to things like *parallel lines*, *perpendicular lines*, and the intersections between them. This gives us another definition of angle:

Definition 3. *An angle is the space formed between two segments of intersecting lines.*

In school we work with many combinations of such intersecting lines, including parallel ones; but the the most basic case is embodied in Figure 7.

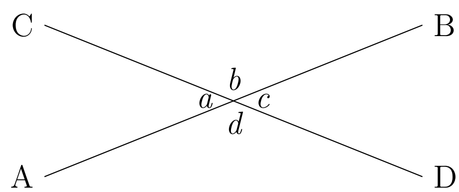


Figure 7: *The most basic intersection definition angle.*

Figure 7 on page 10, obviously, represents two intersecting lines, AB and CD. It also shows angles a, b, c, and d. This is the most basic, the most fundamental, of all possible sets of angles. Since our criterion throughout this exercise is *practical use*, let us judge the divisions of the circle as regards this definition of angle by some practical test: given the value of one of these angles, how do we arrive at the values of the other three?

(Notably, it's clearly possible, even trivial, to do so with any of the four systems being examined; our question is which system makes this task *easiest*.)

At first glance, it appears that what we're dealing with here is really a full circle: $a + b + c + d = \varsigma$. Therefore, it appears that a division into one dozen parts will be most advantageous. However, the question really isn't that simple.

Figure 8 on page 11 shows the steps necessary to solve this basic problem.

The clear winner here is the division into two dozen; there is no necessity for any multiplication or division, and only the number 1 intrudes. A surprise contender is the division into four dozen, which is a close player with the double dozen. This four dozen division requires only a factor of two, less simple than a factor of one but only marginally so. Three dozen is, as

Two dozen

1. a is one third of a full circle; since u is half a circle, $a = 2u/3$, which of course is 0;8.
2. Opposite angles are equal; so given that $a = 0;8$, we know that $c = 0;8$.
3. $a+b = u$, obviously. So $b = u-a$.
4. $u = 1$ and $a = 0;8$, so $b = 1 - 0;8$.
5. $b = 0;4$.
6. Opposite angles are equal; so $d = b$.
7. $a = 0;8; b = 0;4; c = 0;8; d = 0;4$.

One dozen

1. a is one third of a full circle; since u is a full circle, $a = u/3$, which of course is 0;4.
2. Opposite angles are equal; so given that $a = 0;4$, we know that $c = 0;4$.
3. $a + b = u/2$. So $b = (u/2) - a$.
4. $u = 1$ and $a = 0;4$, so $b = 1/2 - 0;4$.
5. $b = 0;6 - 0;4$.
6. $b = 0;2$.
7. Opposite angles are equal; so $d = b$.
8. $a = 0;4$; $b = 0;2$; $c = 0;4$; $d = 0;2$.

Three dozen

1. a is one quarter of a full circle; since u is a third of a circle, $a = 0;9$.
2. Opposite angles are equal; so given that $a = 0;9$, $c = 0;9$.
3. $a + b = 3u/2$. So $b = (3u/2) - a$.
4. $u = 1$ and $a = 0;9$, so $b = (3/2) - 0;9$.
5. $b = 1;6 - 0;9$.
6. $b = 0;9$.
7. Opposite angles are equal; so $d = b$.
8. $a = 0;9$; $b = 0;9$; $c = 0;9$; $d = 0;9$.

Four dozen

1. a is one eighth of a full circle; since u is a quarter of a circle, $a = 0;6$.
2. Opposite angles are equal; so given that $a = 0;6$, $c = 0;6$.
3. $a + b = 2u$. So $b = 2u - a$.
4. $u = 1$ and $a = 0;6$; so $b = (2 \cdot 1) - 0;6$.
5. $b = 2 - 0;6$.
6. $b = 1;6$.
7. Opposite angles are equal; so $d = b$.
8. $a = 0;6$; $b = 1;6$; $c = 0;6$; $d = 1;6$.

Figure 8: *Steps for extrapolating three angles from one angle given two intersecting lines; the intersection angle definition.*

expected, a disaster; and one dozen, despite its outward simplicity, adds the complication of dividing by two, and the only thing worse than multiplying by two is dividing by it.

Final verdict: two dozen, 1; four dozen, 2; one dozen, 3; three dozen, 4.

2.4 Circular Definition—2, 1, 4, 3

Now we come to what I have called the *circular definition*, so called simply because it is based on the circle:

Definition 4. *An angle is some portion of a turn about the circle.*

Figure 9 on page 12 shows the results of these divisions of the circle; the results are somewhat suprising.

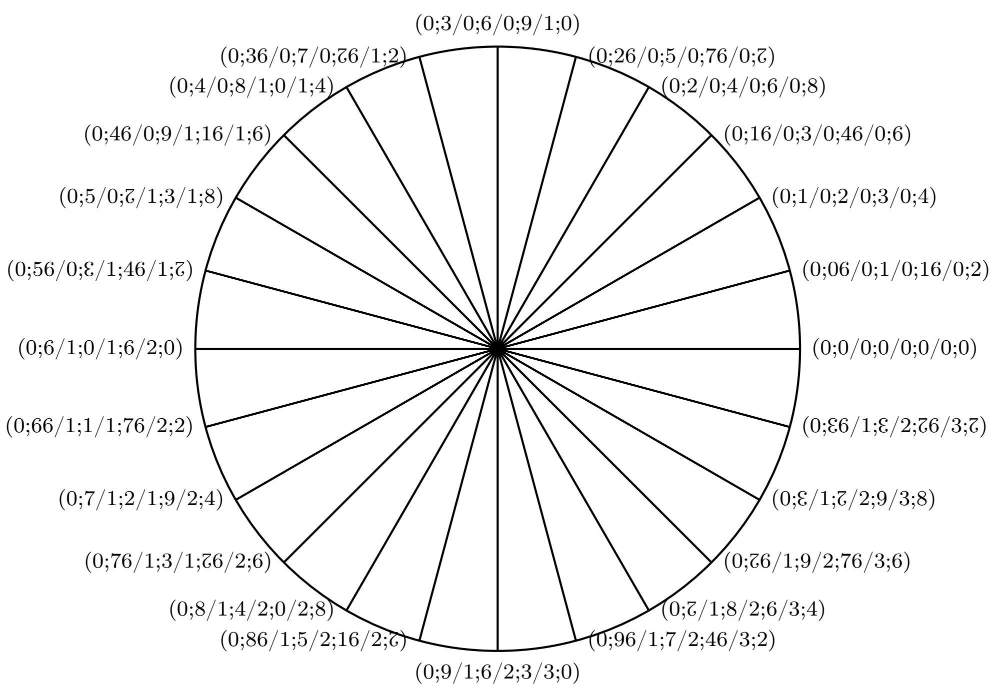


Figure 9: *The circle divided into fifteen-degree increments.*

It might be questioned whether a division into fifteen-degree increments is really a fair comparison. The response to that question is thus:

1. These angles are incredibly common, as anyone who has done any significant work in geometry or trigonometry knows.
2. The binary divisions of the circle are vitally important. These angles encompass three such binary divisions (one hundred and eighty degrees; ninety degrees; and forty-five degrees).
3. The ternary divisions of the circle are vitally important. These angles encompass that division (one hundred and twenty and two hundred and forty degrees).

Still, lest anyone claim that the fifteen-degree increments is a selection bias toward the division into two dozen, we can prepare a table comparing the angles of repeated bisections and trisections of the circle, along with these fifteen-degree increments, and determine which division produces the simplest angular measurements.

Figure ↊ on page 14 shows these prominent angles all charted together. The angles which are merely fifteen-degree increments and not otherwise important are displayed in normal type; the bisections of the circle are displayed in boldface type; and the trisections of the circle are displayed in italic type. Bisections of the trisection are displayed in bold italic.

Since the second half of the circle is essentially a repeat of the first half, to manage chart size most of these angles are shown only in the first half.

Before preparing this chart, my presumption was that the division into one dozen parts would easily take the prize in this section. However, upon closer examination of all the angles, the answer isn't so clear.

The division of the circle into one dozen parts is superficially a very intuitive way to ensure that all the primary angles are simple fractions. A quarter of a circle is 0;3, one quarter; a third of a circle is 0;4, one third; and so forth. However, we often work in much finer gradations of the circle than that; and really, the parts of the circle when it is divided into one dozen parts are simply too large.

The division into two dozen parts makes each fifteen-degree increment into a single-digit uncia; with the division into one dozen parts, every other such division must go into bicias to be accurately stated. These very important and frequently-used angles are thus better handled by a division into two dozen, rather than one dozen, parts.

Bisections of the circle quickly begin to favor the division into two dozen, as well; on only the third bisection (forty-five degrees), one dozen must go into two digits; and on the fourth it must go into *three*, while two dozen happily remains at two.

Deg. (Dec.)	One Doz.	Two Doz.	Three Doz.	Four Doz.
0	0	0	0	0
5.625	**0;023**	**0;046**	**0;068↊**	**0;09**
11.25	**0;046**	**0;09**	**0;1157**	**0;16**
15	0;06	0;1	0;16	0;2
20	0;09	0;14	0;2	0;28
22.5	**0;09**	**0;16**	**0;22↋1**	**0;3**
30	0;1	0;2	0;3	0;4
40	*0;14*	*0;28*	*0;4*	*0;54*
45	**0;16**	**0;3**	**0;46**	**0;6**
60	***0;2***	***0;4***	***0;6***	***0;8***
75	0;26	0;5	0;76	0;↊
80	*0;28*	*0;54*	*0;8*	*0;↊8*
90	**0;3**	**0;6**	**0;9**	**1;0**
105	0;36	0;7	0;↊6	1;2
120	*0;4*	*0;8*	*1;0*	*1;4*
135	0;46	0;9	1;16	1;6
150	0;5	0;↊	1;3	1;8
165	0;56	0;↊	1;46	1;↊
180	**0;6**	**1;0**	**1;6**	**2;0**
195	0;66	1;1	1;76	2;2
210	0;7	1;2	1;9	2;4
225	0;76	1;3	1;↊6	2;6
240	*0;8*	*1;4*	*2;0*	*2;8*
255	0;86	1;5	2;16	2;↊
270	0;9	1;6	2;3	3;0
285	0;96	1;7	2;46	3;2
300	0;↊	1;8	2;6	3;4
315	0;↊6	1;9	2;76	3;6
330	0;↋	1;↊	2;9	3;8
345	0;↋6	1;↋	2;↊6	3;↊
360	1;0	2;0	3;0	4;0

Figure ↊: *Angular measurements as portions of a circle according to the four divisions.*

Trisections of the circle are effectively a draw between the two. Even on the second trisection, and on the bisection of the first trisection, the number of digits in the two divisions are equivalent.

The trisections are the first area which really favor the division into three dozen parts; we see these plainly charted out in simple, single-digit uncias. On the other hand, the bisections by and large prove terribly awkward; there is little else to be said for dividing the circle into three dozen parts here.

The division into four dozen is excellent, and in this case excels both one and two dozen in terms of circular bisections, remaining at only two digits all the way to the sixth bisection. It also matches one and two dozen in the trisections. On the other hand, its gradations are really too small; for each increment of fifteen degrees, they progress by two uncias, which is a finer gradation than will normally be needed. For this reason, it winds up third.

This result is, in retrospect, unsurprising. The division into two dozen parts is almost universally observed throughout the world, and there is no compelling reason not to observe this division (as there is with the almost universal practice of decimal arithmetic). Longitudes are based on the *hour* (and also provide our time zones), and thus divide the earth into two dozen parts. Astronomers read the sky in terms of *hours* of right ascension, and thus divide the sky into two dozen parts (or rather, the half of it that we can see into one dozen parts, which amounts to the same thing).

Given this chart, it is unsurprising that the division of the circle into two dozen parts is so universal; it's much easier to use circles this way. The dozenal system is all it needs to perfect it.

Final verdict: two dozen; one dozen; four dozen; three dozen.

3 Classical Geometry

Having investigated *what angle is*, we can now proceed to *what angles are used for*. The first field we will investigate along these lines will be *geometry*, the mathematics of shape, size, and space. In reference to geometry, we will consider five primary branches of inquiry, corresponding to important branches of mathematics learned by children: parallel lines, polygons, circles, polyhedra, and spheres.

Importantly, in geometry the right angle is really the most important for *construction*; however, since we are here concerned with calculation as well, we will be proceeding with no such assumptions.

3.1 Parallel Lines—3, 1, 4, 2

Two lines are said to be *parallel* when, if extended infinitely in both directions, they will never meet in either direction. When we throw another line across these parallel lines, important properties begin to emerge. The basic situation is something like Figure Ɛ on page 16.

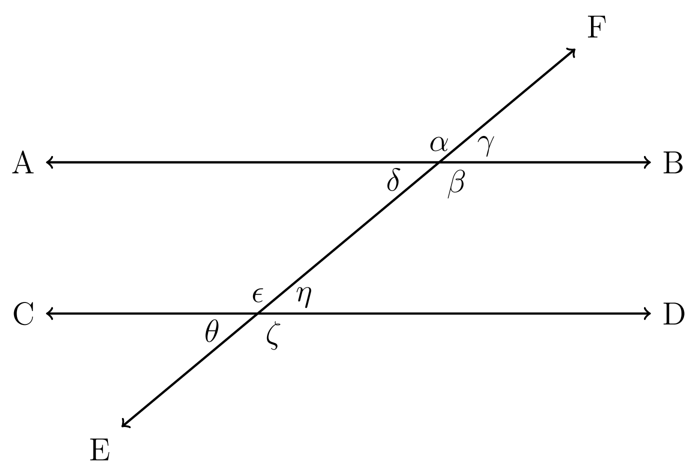


Figure Ɛ: *Two parallel lines cut by another nonparallel line and the angles resulting therefrom.*

From these three lines—two parallel lines and another cutting through them, called the *transversal*, here line EF—follow a number of postulates, most of which we're all quite familiar with:

1. Opposite angles are equal; that is, angles produced by the intersection of two lines are equal on both sides of that intersection. In this example, $\alpha = \beta$, $\gamma = \delta$, $\epsilon = \zeta$, $\eta = \theta$.
2. Corresponding angles are equal; that is, angles which are on the same side of the transversal, and on the same side of their line, but on different parallel lines, are equal. In this case, $\gamma = \eta$, $\alpha = \epsilon$, $\theta = \delta$, $\zeta = \beta$.
3. Alternate interior angles are equal; that is, angles in between the parallel lines, but on opposite parallel lines, and on opposite sides of the transversal, are equal. In this case, $\delta = \eta$, $\beta = \epsilon$.
4. Alternate exterior angles are equal; these are analogous to alternate interior angles, but on the outside of the parallel lines. In this case,

$\alpha = \zeta$, $\gamma = \theta$.

5. The sum of consecutive interior angles is equal to two right angles; or, alternatively, one semicircle; or, alternatively, half of one circle. In this case, $\delta + \epsilon = \frac{c}{2}$; $\beta + \eta = \frac{c}{2}$.
6. The sum of consecutive exterior angles is equal to two right angles; or one semicircle; or half of one circle. In this case, $\gamma + \zeta = \frac{c}{2}$; $\alpha + \theta = \frac{c}{2}$.

These rules mean that if any one of the these angles, α through θ, are known, all the other angles can be extrapolated from it. Assume, for example, that $\alpha = \frac{c}{3}$. From that we know that β, the opposite angle; ϵ, the corresponding angle; and ζ, the alternate exterior angle, are all also equal to $\frac{c}{3}$. From these, we can determine that $\gamma = \left(\frac{c}{2} - \frac{c}{3}\right)$, or $\frac{c}{6}$; we can also determine that $\delta = \frac{c}{6}$. From these, we can determine that $\eta = \theta = \frac{c}{6}$, and we have proven the sides of all angles from knowing only a single one.

The key to this, however, is that certain angles *sum to be equal to one hundred and eighty degrees*; that is, to one half of a circle. This is what makes such feats of extrapolation possible. This is a great win for the division of the circle into two dozen parts. It is true, of course, that the angles surrounding a point add up to a single circle; however, this is of little help in these situations, as one must first divide it into two before one can extrapolate any further angles from it.

As usual, it is easier to multiply than divide; so four dozen parts takes second place. This is followed by one dozen parts, with three dozen parts again taking last.

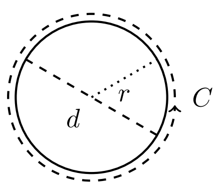
3.2 Circles

A *circle* is defined most simply as a curve closing on itself which is always the same distance from a single central point. A line drawn from the center to any point on its edge is always the same length, and is called the circle's *radius*; a line drawn through the center from edge to edge is called the circle's *diameter*, and is equal to twice the radius. The length of the edge of the circle is called its *perimeter*.

3.2.1 Perimeters of Circles—1, 2, 4, 3

The perimeter of the circle, and its relationship to the size of the circle, has been a matter of fascination for geometers for thousands of years. The bottom line is that the relationship between the perimeter of a circle, or any part thereof, and the radius or diameter of that circle is a mysterious number, irrational, infinite and nonrepeating, and destined to show up in many different areas of mathematics.

Figure 10 on page 18 demonstrates these relationships in light of our current issue. The circle graphically portrays the circumference (C), radius (r), and diameter (d) of the circle, and the table on the right shows the circle constants (as seen above, the number of radians present in the angular unit) defined first in terms of the radius and diameter *and* in terms of the full circle, then in terms of the radius and diameter *and* in terms of itself. As usual, u indicates the angular unit: a full circle for the division into one dozen, a half circle for the division into two dozen, and so forth.



	Full Circle		Itself	
Division	Radius	Diameter	Radius	Diameter
One Dozen	$\varsigma = \frac{C}{r}$	$\varsigma = \frac{2C}{d}$	$\varsigma = \frac{u}{r}$	$\varsigma = \frac{2u}{d}$
Two Dozen	$\pi = \frac{C}{2r}$	$\pi = \frac{C}{d}$	$\pi = \frac{u}{r}$	$\pi = \frac{2u}{d}$
Three Dozen	$\psi = \frac{C}{3r}$	$\psi = \frac{2C}{3d}$	$\psi = \frac{u}{r}$	$\psi = \frac{2u}{d}$
Four Dozen	$\eta = \frac{C}{4r}$	$\eta = \frac{C}{2d}$	$\eta = \frac{u}{r}$	$\eta = \frac{2u}{d}$

Figure 10: *Perimeter of a circles and constants associated therewith.*

In discussions such as this, it will often be pointed out that only ς can be defined as a simple ratio in relation to the *radius*, without any other factor; and that the radius is much more important in most branches of mathematics than the diameter is. The second of these statements is certainly true; the radius is more important in most branches of mathematics than the diameter is. But as Figure 10 shows, the first statement—that only ς can be defined as a simple ratio with the radius without additional factors—is true *only to an extent*.

That is, it's true only to the extent *that we defined the constant in terms of a full circle*. As Figure 10 clearly shows, *all* of these constants, and really an infinite number of other possible constants, can be defined in terms of a simple ratio with the radius *when defined in terms of themselves*.

Let us take four dozen as an example. Assume that we have adopted an angular unit η, defined as the number of radians in a single right angle.

(That's about 1;6↊24 radians.) Given that this angle equals 1;0 in this system (since it's our unit of angle), we can prove that it equals a simple ratio in terms of its radius by simple algebra:

$$C = \eta 4r$$
$$\frac{C}{4} = \frac{\eta 4r}{4}$$
$$\frac{C}{4} = \eta r$$

But we've already seen that $\frac{C}{4}$, the right angle, is our unit of angle; so:

$$u = \eta r$$
$$\eta = \frac{u}{r}$$

Furthermore, we've already seen that u will be equal to 1;0; so:

$$\eta = \frac{1}{r}$$
$$\eta r = 1 = u$$

The same algebra can be done for any of the other proposed units of angle. In other words, our unit of angle will *always* equal the number of radians in that unit of angle multiplied by the radius; that is, equal-length arms forming the angle in question.

Thus, to say that the full-circle unit has some advantage here is simply exposing a circle bias; considered on their own terms, each unit of angle has the same convenient relationship to the radius. The full-circle unit only has an advantage if we assume that the full circle is the basic angular unit, which is precisely the question we're trying to answer here. Therefore, such an assumption is unwarranted.

However, Figure 10 does show us something else: namely, that the full-circle unit has an advantage *when calculating the perimeter of a circle*. For this particular task, assuming that the radius is known, the full circle has the advantage of being a simple ratio with the radius.

On the other hand, it is a difficult question to determine whether this means the full-circle unit should win this category, because the division of the circle into two dozen *also* has a simple relationship with a fundamental measurement of the circle; namely, the diameter. Whether the radius or diameter is more fundamental is immaterial here; the question is whether this particular task is easier to accomplish with one or the other angular unit.

The real difficulty in determining circumferences is in finding the actual center; this is a non-trivial task. Typically, though, the best way would be to place a compass at the best estimate of the center, attempt to draw the circle, and move the point of the compass until it traces out the same curve as the circle in question. Once this has been done, the radius is easily known; therefore, it seems that the full-circle unit is the easiest for accomplishing this task. Indeed, using ς for this task makes all the fractions of the perimeter of a circle equal to the fractions of ς number of radians: a half a circle is $\frac{\varsigma}{2}$, a quarter $\frac{\varsigma}{4}$, and so forth. This means that the numeric value of the perimeter of a unit circle, or any fraction thereof, is equal to the number of radians in its subtended angle, which is no small convenience. All the other units must introduce some scaling factor to make this work.

On the other hand, the diameter of a circle, rather than its radius, can more easily be determined simply by running a straightedge down the circle, allowing its edge to be a chord (a straight line touching the circumference in two and only two places); the longest such chord is the diameter. This is a much simpler and less error-prone procedure than the easiest means of finding a center and a radius.

Still, in most applications the circle is being constructed on a known radius; this is, therefore, a resounding win for the division into one dozen.

The other rankings fall into place in an easily observed manner.

3.2.2 Areas of Circles—1, 2, 4, 3

Areas of circles are less clear than perimeters. The area of a circle has traditionally been calculated according to a simple formula:

$$A = \pi r^2$$

π being the number of radians in a half-circle, it may seem clear that we can simply grade this category and move on. Indeed, as Figure 11 on page 1↋ demonstrates, π has a real way of making the area of circles seem quite simple.

Figure 11 indicates that π, and therefore the division into two dozen parts, enjoys the same superiority in areas that ς, and with it the division into one dozen parts, enjoys with circumferences.

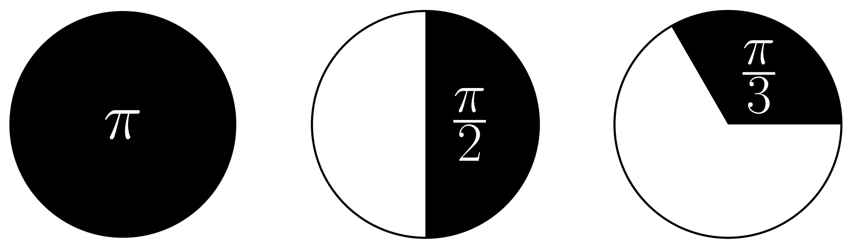


Figure 11: *Areas of circles and fractions thereof expressed as π and fractions thereof.*

Partisans of ς, however, will point out that the situation is not so simple. The area of the circle is, they argue, simply a special case; the general case is the area and volume of an n-sphere. This argument can get very complicated; it asserts that ς is the proper constant for determining circular shapes in any number of dimensions, and that it only happens to be divided by two for the area of the circle in two dimensions. In other words, the claim is that the number π in this formula is simply an accident, and that using π rather than $\frac{\varsigma}{2}$ obscures the real mathematical relationships of circular shapes in n dimensions.

However, given that ς is the ratio of a full circle to its radius, it's unsurprising that it will have fewer factors. The formula for ς for the surface and volume of an n-sphere is the following:

$$S_n = \left(\frac{\varsigma^{\lfloor \frac{n}{2} \rfloor}}{(n-2)!!} (n \bmod 2 + 1) \right) r^{n-1}$$

$$V_n = \left(\frac{\varsigma^{\lfloor \frac{n}{2} \rfloor}}{n!!} (n \bmod 2 + 1) \right) r^{n}$$

But it's an open question whether this is really any simpler than the other constants *for the most common applications*. Calculating the four-dimensional surface area of a 4-sphere is an unusual job; but in two and three dimensions, the calculations must be done quite frequently. We can get the same equations in terms of π as follows:

$$S_n = \left(\frac{2^{\lfloor \frac{n}{2} \rfloor} \pi^{\lfloor \frac{n}{2} \rfloor}}{(n-2)!!} (n \bmod 2 + 1) \right) r^{n-1}$$

$$V_n = \left(\frac{2^{\lfloor \frac{n}{2} \rfloor} \pi^{\lfloor \frac{n}{2} \rfloor}}{n!!} (n \bmod 2 + 1) \right) r^n$$

In general form, these are certainly more complex. But it's worth noting that with *even* values of n, the term of $2^{\lfloor n/2 \rfloor}$ will cancel out with the denominator, producing often quite simple equations, including a number that we've already noticed:

$$S_2 = \left(\frac{2^{\lfloor \frac{2}{2} \rfloor} \pi^{\lfloor \frac{2}{2} \rfloor}}{(2-2)!!} (2 \bmod 2 + 1) \right) r^{2-1} = \frac{\pi}{1}(1+1) r^1 = 2\pi r$$

$$V_2 = \left(\frac{2^{\lfloor \frac{2}{2} \rfloor} \pi^{\lfloor \frac{2}{2} \rfloor}}{2!!} (2 \bmod 2 + 1) \right) r^2 = \frac{\pi}{1}(1+1) r^2 = \pi r^2$$

These are, of course, the formulas which give the circumference of the circle (the surface of a 2-sphere) and the area of a circle (the volume of a 2-sphere). Moving on into three dimensions, we get similarly easy formulas:

$$S_3 = \left(\frac{2^{\lfloor \frac{3}{2} \rfloor} \pi^{\lfloor \frac{3}{2} \rfloor}}{(3-2)!!} (3 \bmod 2 + 1) \right) r^{3-1} = \left(\frac{2^1 \pi^1}{1} \cdot 2 \right) r^2 = 4\pi r^2$$

$$V_3 = \left(\frac{2^{\lfloor \frac{3}{2} \rfloor} \pi^{\lfloor \frac{3}{2} \rfloor}}{3!!} (3 \bmod 2 + 1) \right) r^3 = \left(\frac{2^1 \pi^1}{3} \cdot 2 \right) r^3 = \frac{4\pi r^3}{3}$$

These are the formulas for the surface area and the volume of a normal sphere (a "3-sphere") respectively.

ς produces some easy formulas here, too:

$$S_2 = \left(\frac{\varsigma^{\lfloor \frac{2}{2} \rfloor}}{(2-2)!!} (2 \bmod 2 + 1) \right) r^{2-1} = \left(\frac{\varsigma^1}{1} (0+1) \right) r^1 = \varsigma r$$

$$V_2 = \left(\frac{\varsigma^{\lfloor \frac{2}{2} \rfloor}}{2!!} (2 \bmod 2 + 1) \right) r^2 = \left(\frac{\varsigma}{2} (0+1) \right) r^2 = \frac{\varsigma r^2}{2}$$

$$S_3 = \left(\frac{\varsigma^{\lfloor \frac{3}{2} \rfloor}}{(3-2)!!} (3 \bmod 2 + 1) \right) r^{3-1} = \left(\frac{\varsigma^1}{1} (1+1) \right) r^2 = 2\varsigma r^2$$

$$V_3 = \left(\frac{\varsigma^{\lfloor \frac{3}{2} \rfloor}}{3!!} (3 \bmod 2 + 1) \right) r^3 = \left(\frac{\varsigma^1}{3} (1+1) \right) r^3 = \frac{2\varsigma r^3}{3}$$

The first of these, the formula for the circumference of a circle, is a wonder of simplicity; the rest are comparably complex with those produced by using π.

The argument is often that using ς elucidates the relationships here, and that may or may not be true; the issue we are addressing here is *ease of use in practical application*, and in this particular section it is *ease of use in calculating areas of circles.*

Furthermore, it's not entirely clear that using ς actually does elucidate these relationships. While it's certainly true that ς is used in the derivation of these formulas, it does not necessarily follow that using ς makes these formulas clearer. We can, in fact, by using the gamma function, derive formulas for the surface and volume of an n-sphere which are written more simply with π:

$$V_n = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

We can then take this basic formula, using π, and differentiate it to produce another formula, which can utilize π or be rewritten to utilize ς:

$$S_{n-1} = \frac{\pi^{\frac{n}{2}} n r^{n-1}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\varsigma^{\frac{n}{2}} r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}$$

However, lest we be accused of bias here, we will give ς the benefit of the dispute and grade it with only one point. Because the addition of factors increase with each additional division of the circle, two dozen will get two points; and because even numbers will cancel out more often than odd, we will give four dozen three points.

3.3 Included Angles—3, 1;6, 4, 1;6

Another aspect of circular angles are *included angles*. These are also knonw as *segments* of the circle. A few new terms will be needed here.

A *secant* of a circle is any line which cuts the circle at two points. (If a line cuts the circle at only one point, it is really merely touching it, and is called a *tangent*.) The segment of a secant line which lies within the circumference is called a *chord*.

If two chords intersect one another at the same point on the circumference of the circle, they cut off a *segment* of the circle. This segment is a part of the arc of the circle, which of course subtends a certain central angle of the circle. The intersecting chords also produce a certain angle. The properties of these two angles depend entirely on the right angle and the straight angle.

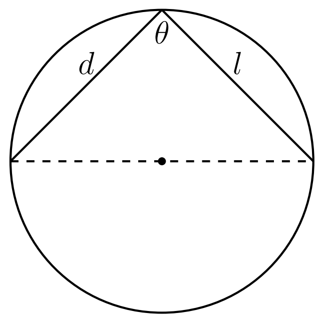


Figure 12: *Segments and angles in a circle.*

Figure 12 on page 22 demonstrates these properties. The angle θ can be *at most* a straight angle; in this case, the lines d and l forming it will be *tangents* rather than *chords* (which, remember, are simply segments of secants). In that event, d and l subtend no arc, because they only intersect the circle at one point.

If, however, θ is *greater* than a right angle but still less than a straight angle, then the arc it subtends will be *greater* than a semicircle. If θ is *less* than a right angle, then the arc it subtends will be *less* than a semicircle. If θ is *equal* to a right angle, then the arc it subtends will be *equal* to a semicircle. In all these cases, of course, if we move to the center and draw l and d from the center to the same points on the circumference, the central angle will be equal to the arc which θ subtends.

These very interesting properties are, of course, entirely dependent upon the right angle and the straight angle; or the angles produced from dividing the circle into four and two parts, respectively. Points are therefore divided accordingly.

3.4 Polygons

A polygon is a two-dimensional shape, bounded by straight lines in a closed chain. Many of these shapes are familiar to us: triangles, squares, pentagons. The lines which bound a polygon are called its *edges* or *sides*; the points where they meet are *vertices* (singular *vertex*). A polygon has a *perimeter* (the distance around its sides) and an *area* (the space within its sides).

A polygon may be *regular*, which means that all of its sides are the same size. It may be *convex*, which means that any lines drawn through the polygon, not tangent to an edge or corner, meets its boundary twice. A *simple* polygon is one which has sides which do not cross one another. A *concave* polygon is neither simple nor convex. There are also *star-shaped* polygons, *star* polygons, *self-intersecting* polygons, and others.

For simplicity's sake, we will be dealing primarily with regular polygons; these are the ones most frequently encountered.

3.4.1 Sum of Interior Angles—3, 1, 4, 2

The interior angles of a polygon have a number of properties which are relevant to our discussion. For example, the interior angles of a polygon are, except for some unusual cases, always *less than a straight line*. Furthermore, the sum of the interior angles of a simple polygon are easily predictable by an equation commonly written thus:

$$\Sigma = (n-2)\left(\frac{c}{2}\right)$$

This equation holds true because the sum of the interior angles of a triangle is one hundred eighty degrees, and any simple polygon can be decomposed into $(n-2)$ triangles.

This bears some additional explanation. Assume a regular polygon with n sides. Choose one vertex of that polygon—any vertex will do—and draw a line connecting the vertex to every other vertex of the polygon. This will yield a number of lines equal to the number of sides in the polygon, minus one, because one cannot draw a line from the chosen vertex to itself. Then, subtract the lines to the vertices adjacent to the chosen vertex, because they are lying on top of the polygon's sides. This yields a number of diagonal lines across the polygon equal to the number of sides minus three. There are one more triangles than that now drawn inside the polygon, making the number of triangles within the polygon equal to the number of sides minus 2. The sum of the internal angles of a triangle is $\frac{c}{2}$ (half a circle); therefore, the number of sides minus two, times $\frac{c}{2}$, equals the sum of the polygon's interior angles.

Doing this manually once is enough to firmly impress it in the mind forever. The operation is depicted in Figure 13 on page 24. There, we have a six-sided polygon; we clearly see that only three diagonal lines are possible from the chosen vertex, which is $n - 3$ (these are shown with dotted lines). We also clearly see that these dotted lines form four triangles; that is, $n - 2$ triangles. Since the sum of the interior angles of each of those $n - 2$ triangles is $\frac{c}{2}$, the sum of the polygon's interior angles is $\frac{c}{2}(n - 2)$.

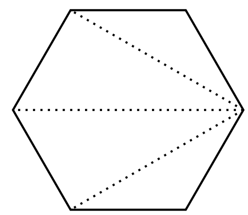


Figure 13: *Decomposition of a polygon into its component triangles.*

That being explained, to compare our various systems, we will look at this equation according to the different divisions.

In Figure 14 on page 24, the sum of the internal angles of the polygon is represented by Σ, an upper-case sigma; the unit for measuring angles is represented by u; and the number of sides in the polygon is represented by n.

One Dozen $\Sigma = (n-2)(\frac{u}{2})$	*Two Dozen* $\Sigma = (n-2)u$
Three Dozen $\Sigma = (n-2)(\frac{3u}{2})$	*Four Dozen* $\Sigma = (n-2)2u$

Figure 14: *Formulae for the sum of interior angles in a regular polygon.*

This plainly shows that the simplest formula is produced by the use of the division of the circle into two dozen parts. Therefore, two dozen takes one point. It is easier to multiply than to divide, so four dozen takes two points. One dozen takes three, and the gnarly fraction in three dozen leads it to take its customary four.

3.4.2 Individual Interior Angles—3, 1, 4, 2

Assuming that the polygon is regular, any interior angle can also be predicted by formula based on the number of sides. The general formula is

$$\theta = \frac{(n-2)\left(\frac{c}{2}\right)}{n}$$

This, like our last formula, gives us the size of the individual angle in *radians*. Figure 15 on page 25 shows the various formulas for our divisions; each of these use θ, lowercase theta, for the angle, u for the circle unit, and n for the number of sides.

One Dozen	*Two Dozen*
$\theta = \frac{u(n-2)}{2n}$	$\theta = \frac{(n-2)u}{n}$
Three Dozen	*Four Dozen*
$\theta = \frac{3u(n-2)}{2n}$	$\theta = \frac{(n-2)2u}{n}$

Figure 15: *Formulae for individual interior angles in a regular polygon.*

This result leads us in the same direction as our last; two dozen takes one point, four takes two, one takes three, and three takes four.

3.4.3 Individual Exterior Angles—1;6, 1;6, 4, 3

The exterior angles of a polygon are those that are *outside* the edges of the polygon at its vertices. These angles are *supplementary* to the interior angles of the polygon:

Definition 5. *Supplementary angles are those which, when added together, total a straight line.*

(Peripherally, we may note here that this is another vitally important concept that centers on the straight line.)

The commonly stated rule regarding the exterior angles of a polygon is that they sum to the same angle as that present in a full circle. However, this is not really true. Each interior angle has two supplements, one by extending one of the sides which meet to form that angle and one by extending the other. This is demonstrated in Figure 16 on page 26.

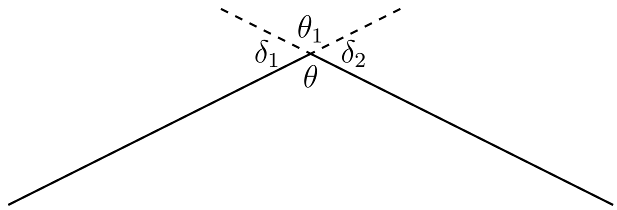


Figure 16: *Measuring an individual exterior angle of a polygon.*

In Figure 16, we see that the interior angle θ has three associated exterior angles: θ_1, its opposite angle; δ_1, its first supplementary angle; and δ_2, its second supplementary angle. Clearly, then, given this, an individual exterior angle of a polygon is equal to

$$\theta_1 + \delta_1 + \delta_2$$

Or, more simply,

$$c - \theta$$

In other words, this is simply the reflex angle of the interior angle; and using our formula from our study of interior angles, it is equal to

$$c - \left(\frac{(n-2)\left(\frac{c}{2}\right)}{n} \right) = \frac{c(n+2)}{2n}$$

where n is the number of sides of the polygon. If we rephrase the equation in terms of $\frac{c}{2}$ (which we will call u), then an individual exterior angle of a polygon is

$$\frac{u(n+2)}{n}$$

This is a lovely formula, and it's hard to see how it could be any simpler; and furthermore, it is clearly related to the formula for an individual *interior* angle that we met earlier. Clearly, then, determining the measure of these exterior angles is another victory for the division of the circle into two dozen parts.

However, a different, less obvious definition for an *exterior angle* has become the norm, one which is not at all obvious considering what we are looking at (as visually demonstrated in Figure 16 on page 26). Rather than considering the reflex angle of θ, the exterior angle is instead considered to be *either* δ_1 *or* δ_2 *but not both*, with θ_1 being ignored. As a result, the exterior angle is defined as the supplementary angle of an interior angle; and the sum of all of these angles in a regular polygon is equal to the angle of a full circle.

The justification for this procedure is generally that, if one begins traversing the outside of an arbitrary edge of the polygon in one direction, this is the sum of the angles one will turn when reaching one's starting point. However, this isn't particularly helpful information by and large; and furthermore, one must arbitrarily choose which direction one is traversing (else one will be forced to sum up *all* the supplementary angles of the interior angles of the polygon, not merely half of them, and come up with a total of $2c$ rather than c), and one must further ignore two important angles (the opposite of the interior angle and one of its supplements) while one is doing so.

Given this rather tortured definition of an exterior angle, clearly this is a win for the division of the circle into one dozen. However, even in this case, the division of the circle into two dozen is a strong contender, given that the exterior angle cannot be determined but for the supplementary angles, which are based on the straight angle, which is the division of the circle into two dozen. (We can certainly simply divide c by the number of vertices in the polygon; but that doesn't change the fact that what we're dealing with are supplementary angles.) Given that both quantities are absolutely vital for making these calculations, first place should be divided between them.

The more logical definition of the exterior angle as the reflex of the interior angle will not be graded, as it is rarely used.

While the interior angles are unquestionably the more important quantities, this will be graded with the same weight as all other categories.

Final verdict: 1;6, 1;6, 4, 3.

3.4.4 Sum of Exterior Angles—1;6, 1;6, 4, 3

We must begin here by acknowledging the strange definition of exterior angles that we noticed in the last section. The most logical definition of a polygon's exterior angles—namely, the reflexes of its interior angles—is not the one generally used. However, given this logical definition, our formula for the sum of exterior angles is easily derived from our formula for individual ones. Figure 17 on page 28 shows these formulas in terms of each of our prospective circular divisions, given u as the corresponding angular unit.

One Dozen	Two Dozen
$\Sigma_{ext} = \frac{u(n+2)}{2}$	$\Sigma_{ext} = u(n+2)$
Three Dozen	*Four Dozen*
$\Sigma_{ext} = \frac{3u(n+2)}{2}$	$\Sigma_{ext} = 2u(n+2)$

Figure 17: *Formulae for the sum of exterior angles.*

The simplest formula is evident; and furthermore, it's plainly related to and easily derivable from the formula for the sum of the *interior* angles.

However, given the strange definition of exterior angles that is commonly used, we need to use a different analysis. Assuming that we are summing only *one* of the supplementary angles of the interior angles, because we are going around the polygon in only one arbitrarily chosen direction, then the sum of the exterior angles of the polygon is c. If we sum *all* the supplementary angles of the interior angles of the polygon, then their sum is equal to $2c$. So this definition seems clearly a win for the division into one dozen; however, given that it depends crucially upon supplementary angles, which are based upon the straight angle (or the half circle), the division into two dozen seems entitled to equal points.

Consequently, we grade this category the same way we did the category of individual exterior angles.

3.4.5 Apothems of Polygons—3, 1, 4, 2

The *area* of a polygon is the space enclosed by its sides. Calculating the area is relatively simple, but involves a few new words. The *perimeter* of the polygon is the sum of the length of its sides; the *apothem* of the polygon is a line drawn from its center to the midpoint of one of its sides. Assuming P is the perimeter and a is the apothem, the formula for the area of a polygon is:

$$A = \frac{Pa}{2}$$

This equation shows why the apothem of a polygon is important.

The apothem of a polygon can be calculated, as well, and this will be the subject of this section. Assuming s is the length of a side, n is the number of sides, c is the angle of a full circle, and r is the radius of a circumscribed circle, the apothem is equal to:

$$a = \frac{1}{2} s \tan\left(\frac{\frac{c}{4}(n-2)}{n}\right) = \frac{s}{2\tan\left(\frac{c}{2}/n\right)} = r\cos\left(\frac{c}{2}/n\right)$$

We have here three formulas which all yield the apothem of a regular polygon. All three of these formulas involve two variables, all involve one trigonometric function; however, the first involves seven operations, the second five, and the third only four. So the equations are grouped here in the order of simplicity.

The simplest formula involves the full circle divided by two; this clearly favors the division of the circle into two dozen parts. However, this also requires circumscribing a circle and taking its radius, which adds considerable complexity to the issue.

The next simplest formula also involves division of the full circle into two, which just as clearly favors the divisions of the circle into two dozen parts.

The last involves $\frac{c}{4}$; in other words, division of the circle into four dozen parts. However, this is the most complex of the equations by a significant margin; so while this earns the division into four parts second place, it is insufficient to push it into first.

Final verdict: one dozen, three; two dozen, one; three dozen, four; four dozen, two.

3.4.6 Perimeters of Polygons—3, 1, 4, 2

The perimeters of regular polygons can also be calculated based partly on its angles; specifically, on its number of sides and on its *radius*; that is, the distance from its center to any vertex. (This is also called its *circumradius*, because it is the radius of the polygon's *circumcircle*, which is the circle that passes through each of its vertices.)

Let c equal the angle of a full circle, n equal the number of sides, P equal the perimeter, and r equal the radius; then

$$P = 2nr\sin\left(\frac{\left(\frac{c}{2}\right)}{n}\right)$$

This only applies to *inscribed* polygons, however; that is, those which are drawn inside a unit circle which passes through each of its vertices. *Circum-*

scribed polygons are those which are drawn *outside* of a circle, such that each of its sides is a tangent line to that circle. The formula for the perimeter of these polygons is:

$$P = 2nr \tan\left(\frac{\left(\frac{c}{2}\right)}{n}\right)$$

Here, again, we find that vital angle, $\frac{c}{2}$.

According to the metrics we've become accustomed to by now, the grades are one dozen, 3; two dozen, 1; three dozen, 4; and four dozen, 2.

3.4.7 Areas of Polygons—1;6, 1;6, 4, 3

Areas of polygons are, as already shown, given by the equation $A = \frac{Pa}{2}$. The P, of course, is the perimeter. However, we know how to calculate the perimeter of a polygon. Assuming that it is an *inscribed* polygon:

$$P = 2nr \sin\left(\frac{\left(\frac{c}{2}\right)}{n}\right)$$

If it is a *circumscribed* polygon, of course, simply substitute the tangent for the sine. Assuming, again, an inscribed polygon, we can substitute our formula for the perimeter in this equation and get an equation for the area:

$$A = \frac{a2nr \sin\left(\frac{\left(\frac{c}{2}\right)}{n}\right)}{2} = nra \sin\left(\frac{\left(\frac{c}{2}\right)}{n}\right)$$

There are also methods of deriving the area of a polygon partly or wholly by means of its angles and the number of its sides *without* utilizing its apothem. Indeed, there are a number of ways of doing so. In the following equations, s is the length of a side, c is the angle of a complete circle, n is the number of sides, and θ is the measurement of an internal angle. One significant means of calculating area by using angles is:

$$A = \frac{ns^2}{4} \cot \frac{\left(\frac{c}{2}\right)}{n}$$

Here we see the telltale signs of the division of the circle into two dozen parts once again. Every $\frac{c}{2}$ would be transferred into a "1;0" in an angular measurement system based on that division. Currently, we commonly write this measurement as π, indicating that it does, in fact, equal π radians.

Two more equations for making these calculations are:

$$A = \frac{ns^2}{4} \cot \frac{\theta}{n-2} = n \sin \frac{\theta}{n-2} \cos \frac{\theta}{n-2}$$

More trigonometry, of course, which is unsurprising given the importance of triangles in the calculation of area. θ represents an interior angle of the polygon, which can be almost anything; this initially appears, therefore, not to favor any of the divisions of the circle we're considering. However, we must remember that the interior angles of a polygon are always either $\frac{c}{2}$ or a multiple thereof. This means that these equations, too, favor the division of the circle into two dozen parts.

Finally, if all the sides of a regular unit polygon are unknown, we can generalize the second of the above two equations to find the area:

$$A = n \sin\left(\frac{\left(\frac{c}{2}\right)}{n}\right) \cos\left(\frac{\left(\frac{c}{2}\right)}{n}\right) = \frac{n}{2} \sin \frac{c}{n}$$

While these two equations are functionally equivalent (the second is merely a simplification of the first by means of a trigonometric rule known as product-to-sum formula), the second is clearly simpler and easier to apply. This clearly favors the division of the circle into one dozen parts. However, the fact that the other methods depend upon an interior angle of the polygon, which is determined by a process clearly favoring the division of the circle into *two* dozen parts, as well as the apothem to which the same comment may be made, makes this category a wash between them.

These equations come from the fact that any polygon with n sides can be divided into n isosceles triangles with a top angle of $\frac{c}{n}$. We can then find the area of each triangle with the normal equation for the area of a triangle, $\frac{bh}{2}$. We just substitute in $\frac{1}{2} \sin \frac{c}{n}$ to find the area of one triangle and multiply by n to find the area of the whole polygon.

On the other hand, the equation can also be derived in another way. All regular polygons are constructed of $2n$ right triangles; the upper angle on each of these right triangles is equal to $\frac{c}{2}/n$. We can perform the same substitutions and identities as done above to derive the same formula, $\frac{n}{2} \sin \frac{c}{n}$.

In any case, our concern here is not the *derivation* of the formula, but its *ease of practical use.* Many of these formulas involve $\frac{c}{2}$; this one involves

c. As such, the division into one dozen and two dozen both earn 1;6 points. Four dozen earns three in third place; three dozen resumes its customary last place.

(Incidentally, there are also two convenient formulas utilizing $\frac{c}{2}$ regarding areas of *inscribed* and *circumscribed* circles of regular polygons. For the area of an *inscribed* circle:

$$A = \frac{\left(\frac{c}{2}\right) s^2}{4} \cot^2 \frac{\frac{c}{2}}{n}$$

And for a *circumscribed* circle:

$$A = \frac{\left(\frac{c}{2}\right) s^2}{4} \csc^2 \frac{\frac{c}{2}}{n}$$

These formulas can be very useful for certain applications.)

3.5 Spheres—2, 4, 8, 6

We've already seen how the equations for these things can be derived in Section 3.2.2 on page 1↊; there is no need to repeat that information here. However, because we've derived points for polyhedra, we will derive points here, as well; and although the equations generated are comparably complex for the one and the two dozen, we will again attempt to avoid the accusation of bias by giving the division into one dozen the benefit of the dispute.

Points here are doubled, because we are really addressing two issues: surface area of spheres, and volume of spheres.

3.6 Polyhedra

A *polyhedron* is a polygon extended into three dimensions. These can become quite complex, even more so than polygons; however, for now we will stick with *regular polyhedra*; that is, those polyhedra with all equal sides and all equal angles; and furthermore, with the Platonic solids. Other polyhedra quickly become so complex that ease of use is no longer our primary concern; they will be exclusively the province of experts, who can handle these things however they will.

Before we begin, it's important to note that polyhedra are typically classified by two numbers, written in the form $\{p, q\}$, called the *Schläfli symbol.* p represents the regular polygon which forms the polyhedron's face; so, for example, if $p = 3$ then the polyhedron's faces are equilateral triangles. q is the number of p which surround a single vertex of the polyhedron. So, for example, the regular quadrahedron, or *cube*, is made up of six squares, three being attached to one another at each vertex; a cube's Schläfli symbol is therefore $\{4, 3\}$.

p and q will be used in this way throughout our discussion of polyhedra.

3.6.1 Internal Characteristics of Polyhedra—1;6, 1;6, 4, 3

To address this issue, we'll need to review a few new terms. The *dihedral angle* is the angle between any two faces of the polyhedron. Remember that this is *not* the interior angle at a vertex, merely that between two faces. This can be determined by a simple formula:

$$\sin\frac{\theta}{2} = \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{p}}$$

But remember that π is the number of radians in a straight angle, or a half-circle.

After we solve for this formula, we simply take the arcsine and then double it for the dihedral angle.

We can also determine an interior angle of a polyhedron by formula; however, here we are dealing of *solid angle*. Solid angle is the equivalent of the angles we have been dealing with, called *plane angles*, extended into three dimensions. There is also the three-dimensional equivalent of the radian, the *steradian*; just as the radian is the ratio of the swept-out arc to the radius, the steradian is the ratio of the surface area swept out to the square of the radius. This is, of course, the surface area divided by the square of the radius; in TGM, this is the *quari*, divided by the square of the radius, and the *Surf*, the unit of area, making the *quariSurf*.

Solid angle is typically represented by the symbol Ω ("omega").

The solid angle at the vertex of a Platonic solid can be calculated once the dihedral angle is known. Let θ be the dihedral angle:

$$\Omega = q\theta - (q - 2)\pi$$

Finally, every vertex of a polyhedron involves a single point surrounded by a number of shapes; however, because the shape is convex (that is, it's not flat), it will never equal one full circle, as it would if the shape were simply two-dimensional. The difference between the actual sum of the angles and the full circle is known as the *defect*, and it is calculated by the following formula, letting δ equal the defect:

$$\delta = 2\pi - q\pi \left(1 - \frac{2}{p}\right)$$

Remember that π is the number of radians in a half circle, and 2π is the number of radians in a full circle. So this category is a wash between the division into one and two dozens. Given that four dozen involves fewer fractions, it takes third place.

3.6.2 Surface Areas of Polyhedra—1;6, 1;6, 4, 3

While with polygons we calculate perimeters, with polyhedra we calculate *surface area*, a calculation that is typically fairly simple: simply calculate the area of a given face of the polyhedron, then multiply by the number of faces.

For this discussion, let n be the number of faces of the polyhedron; s the length an edge; S_n the total surface area; r the radius, or the distance from the center of the polyhedron to the vertex; a the apothem (the distance from the center of the polyhedron to the center of one of its faces); A the area of a face; and $k$and V the volume. Given this, the surface area of a polyhedron is given by:

$$S_n = nA$$

Simple as can be. However, since we've already graded determining the area of a polygon, this category will have to graded the same way.

3.6.3 Volumes of Polyhedra—1;6, 1;6, 4, 3

The volume of a polyhedron is a more difficult question; however, we can draw some analogy between this and the area of a polygon.

Just as any regular polygon can be decomposed into its component triangles, so also a regular polyhedron can be decomposed into its component pyramids. Each face of the polyhedron will be the base of one such pyramid, and the center of the polyhedron will be its top point. But the volume of a pyramid is a solved problem, easily given by the following equation, where h is the height of the pyramid and A is the area of the base:

$$V = \frac{hA}{3}$$

Extending this to all polygons, let n be the number of faces:

$$V = n\left(\frac{aS_n}{3}\right)$$

We see that this question is entirely dependent upon *areas*; therefore, this category should be graded just as the area category was.

4 Cartesian Geometry

We now proceed to the consideration of geometry beyond the classical geometry that was known to the ancient Greeks. Here we will consider two essential points: the Cartesian plane, and the geometry of complex numbers.

4.1 The Cartesian Plane—3, 1, 4, 2

We've already seen the number line (in Section 2.2 on page 7); the Cartesian plane is essentially an interesting and extremely useful way to extend the simple number line, which is by nature a one-dimensional construct, into two dimensions.

The number line we met in Section 2.2 simple shows real integers mapped along a line, with negative numbers in one direction and positive numbers in another. If we take *two* number lines, however, and make them perpendicular to one another, then each number line becomes an *axis* (conventionally called the *x-axis*, for the horizontal one, and the *y-axis*, for the vertical one), and we move from a simple number line to what for centuries has been referred to as the Cartesian plane, after Descartes, the first mathematician to do significant work with it.

We can even lay out a grid for each integer (or any other unit, for that matter) on the Cartesian plane, which makes the concept even more useful. Figure 18 on page 34 shows the Cartesian plane.

This simple diagram shows the two axes, the horizontal x-axis and the vertical y-axis intersecting at point 0 on each of their number lines. (Point 0 is unlabelled.) Because we have *two* number lines, we can no longer unambiguously refer to any given point by a single number; instead, we need to

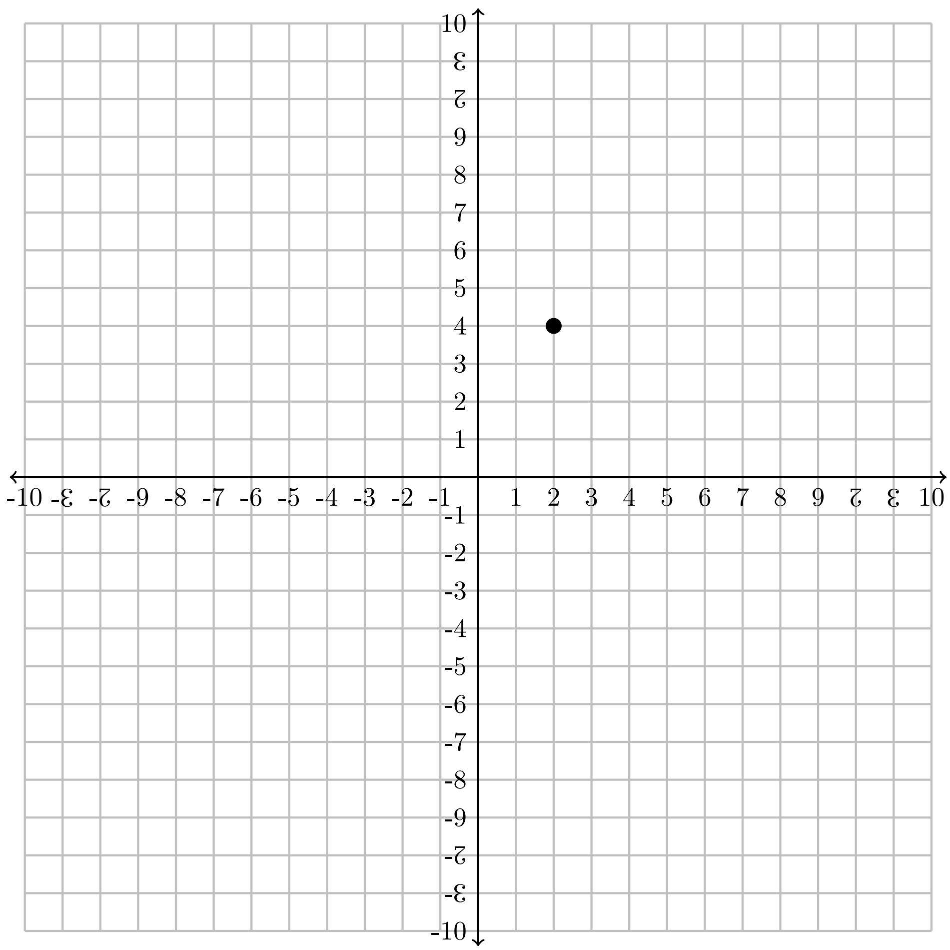


Figure 18: *The Cartesian plane.*

use an *ordered pair*. The first number of such an ordered pair indicates the number from the x-axis; the second indicates the number from the y-axis. This ordered pair is always enclosed in parentheses, and its two members are separated by a comma. So, for example, $(2, 4)$ indicates the point at two on the x-axis and four on the y-axis; Figure 18 shows this point as a black dot. The arrows on either end of the axes indicate that these lines, being number lines, potentially continue to infinity; we've only drawn in certain points, but there are infinitely many points which could be depicted this way.

Of course, we can draw any shape whatsoever on this plane, depicting its points by means of ordered pairs. However, of special note is the *circle*. Let's consider the circle on this plane for a moment; indeed, let's present another figure, a circle drawn on a Cartesian plane. A more simply drawn plane will suffice for depicting this. This will also be a *unit circle*; that is, a circle whose radius is simply one unit. Figure 19 on page 35 depicts such a unit circle.

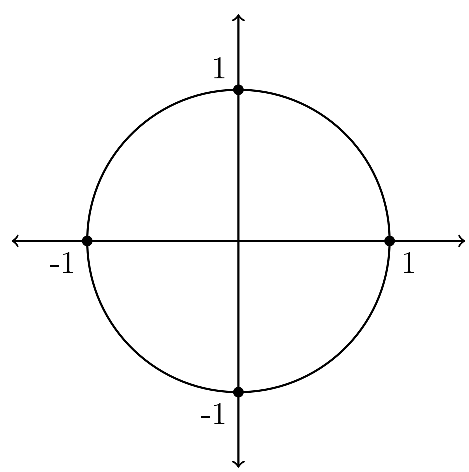


Figure 19: *A unit circle.*

As can be plainly seen, this circle can be described as passing through points (1,0); (0,1); (-1,0); and (0,-1). Now how does this relate to our proposed divisions of the circle being used as the basis for a unit of angle?

Let's examine what we can tell about the angle we're considering based upon changes in coordinates on the Cartesian plane. When we change direction, we wind up at different points along the edge of this circle, according to the formula $x^2 + y^2 = r^2$, where x is the point along the x-axis, y is the point along the y-axis, and r is the radius of the circle. Interestingly, the x coordinate on the unit circle is given by the cosine of the angle we've turned, and the y coordinate is given by the sine of the same angle, assuming that we've started at (1,0); this makes it very easy to compute our point along the edge of the circle, and to determine what kind of change of direction we've undergone.

For example, if we simply reverse our direction while we're standing at our starting point, we have gone from (1,0) to (-1,0). This brings us to the crucial point: *reversal of direction along an axis is the same as negation of the coordinate on that axis.* This is true no matter where we begin; for example, if we begin at (0,-1), and we move to (0,1), we know that we have reversed our direction along the y-axis. If we begin at, for example, (-0;6, 0;↊485), and we've moved to (-0;6, -0;↊485), we know that we've reversed our direction along the y-axis, and are pointed not at one hundred and twenty degrees (our beginning point), but at two hundred and forty degrees, the same x-coordinate but on the opposite side of the y-axis. If, on the other hand, we started at (-0;6, 0;↊485) and moved to (0;6, -0;↊485), we know that we've reversed our direction along *both* axes, and rather than being at one hundred and twenty degrees, or two hundred and forty (if we reversed only along the y-axis), we're at three hundred degrees.

Now we can consider what our four divisions of the circle mean in reference to the Cartesian plane; that is, how transparent they make the relationships between coordinates based upon changes in angle.

Four dozen With this division, the right angle is equal to 1;0, and there is 4;0 in a full circle. To reverse direction, then, we add 2;0 to our angle; to half reverse direction, we add 1;0. So, for example, if our starting point is (-0;6, 0;↊485), our angle is 1;4; to reverse our direction across both axes, we add two, 3;4; to reverse just across the y-axis, we add 0;8 to get to the y-axis (which is equal to 2, of course), then add 0;8 more to get to the opposite angle, 2;8; to reverse direction across the x-axis, we subtract 0;4 to get to the x-axis (which is equal to 1 on this side of the y-axis), then 0;4 more to get to 0;8. While potentially powerful, this method is cumbersome, requiring too much fractional work for convenience; and the units are really too small, requiring us to regularly go over two units for these reversals.

Three dozen As usual, there is little to say for this division of the circle. There is no easy way to reflect across either of the axes or across both;

even to reflect across both, one must add 1;6, doable but cumbersome. Reflecting across the x-axis from the unit itself is easy, 1;0 to 2;0; but from other units it becomes more difficult, 0;8 for example reflecting to 2;4. There is little to recommend this division over our other three.

Two dozen Again as usual, the division into two dozen presents great advantages. To reflect across both axes of the circle, we simply add 1;0. So, for example, (-0;6, 0;Ƨ485) is an angle of 0;8; (0;6, -0;Ƨ485) is an angle of 1;8. *Addition or subtraction of the unit is the same as negating both coordinates.* This melds extremely well with what we've learned about the Cartesian plane. To reflect across the x-axis, we simply "reflect across one"; that is, take the distance between the current angle and one, then add or subtract that to or from 1;0. For example, our angle of 0;8 can be reflected across the x-axis by taking the distance between 0;8 and 1;0, which is 0;4, and adding that to 1;0, giving us 1;4. 1;4 is 0;8 reflected across the x-axis. To reflect 1;4 across the x-axis, do the opposite: the distance from 1;4 to 1;0 is 0;4, so subtract that from 1;0 to get 0;8. We've already seen that negation of coordinates corresponds to addition or subtraction of one; this procedure continues that correspondence. Reflecting across the y-axis is just as easy: simply subtract the fractional part of the current angle from 1;0, and add one if necessary. So 0;8 reflected across the y-axis is 1;0 − 0;8 = 0;4; 1;4 reflected across the y-axis is 1;0 − 0;4 = 0;8, plus 1;0 is 1;8. Again, if negation of coordinates is the same as an addition or subtraction of one, this procedure reflects that by centering on the number 1;0, the very unit of angle.

One dozen This division tends to obscure the relations of the Cartesian plane by showing every angle as a subset of one. Reflection across both axes is a simple addition by one half; (-0;6, 0;Ƨ485) corresponds to an angle of 0;4, and its opposite angle is 0;Ƨ. Reflecting across the x-axis is a matter of "reflecting across one half," similar to the procedure we saw in "reflecting across one" under two dozen; 0;4 plus 0;2 equals one half, which is the x-axis; plus 0;2 more is 0;8, which is the reflection across the x-axis. Easy, but more cumbersome than reflecting across one. Reflection across the y-axis likewise requires a similar procedure: taking the absolute value of the current angle subtracted from 0;6, then adding 0;6 if necessary. Again, doable, but cumbersome; and rather than reversal of direction centering upon the unit itself, it centers upon half the unit. Changing direction by the unit itself is useless; we gain no knowledge, because it's impossible to tell whether we've made a full turn or simply stood still. In the Cartesian plane, only fractional parts of the unit are useful.

All in all, the division into two dozen seems a clear winner in this respect; it most easily maps the dual coordinates of Cartesian geometry into a single-numbered angular unit. After this comes the four dozen; four dozen beats out one dozen due simply to the fact that for one dozen, the unit itself represents essentially standing still, presenting no new knowledge to the user. Following this is the one dozen, with the three dozen taking its usual place at the end.

4.2 Complex Numbers

Complex numbers are rarely used in daily life; however, they do have important applications in several fields, and their geometry will be helpful to us in our study. Therefore, we will consider them briefly here.

4.2.1 Complex Numbers in General—3, 1, 4, 2

Complex numbers are often called *imaginary* numbers, which was originally a derogatory term implying their uselessness. However, they are, in fact, imaginary, insofar as they are not real; that is, the represent numbers that don't really exist. Still, their use makes certain calculations noticeably easier.

It's well-known to most people that we cannot take the square root of a negative number; this is because negative numbers multiplied by negative numbers make positive numbers, so squaring a negative makes a positive. Then, if we take the square root of that positive, we get a positive. On this level—the level of real numbers—the square root of a negative number makes no sense.

However, at some point mathematicians began using the symbol i to help represent these numbers, and found that they had some interesting properties. The formula is:

$$\sqrt{-1} = i$$

There is a full system of rules for arithmetic and algebra involving these numbers, but we need not get too involved in such things here; the important thing is to understand what we're talking about, so that when we graph them we can recognize what we're looking at.

We've already seen how a Cartesian plane is formed from the combination of two number lines (in Section 4.1 on page 33); now we will do the same, but we will map *real* numbers (roughly, the ones we normally use, like -4 and 3) on the x-axis and *imaginary* numbers (like i) on the y-axis. Figure 1↊ on page 39 shows the simplest form of this complex plane, analogous to the unit circle that we saw when discussing the Cartesian plane. Five points have been marked and labeled on it.

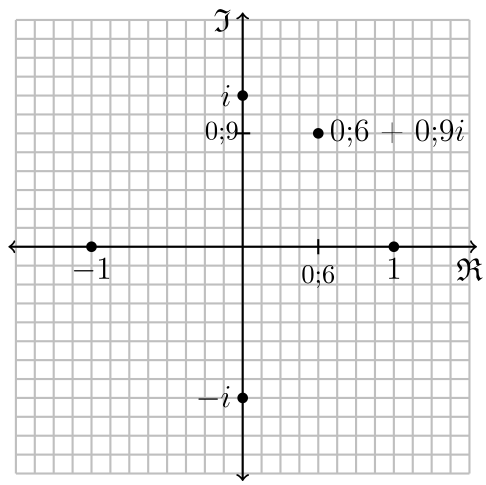


Figure 1↊: *The complex plane.*

Here things are getting interesting. As can be seen here, the transfer from real to imaginary numbers can be visualized as a *rotation*; that is, as a rotation of one right angle, or ninety degrees. A further rotation of one right angle brings us back into real numbers, -1; a further rotation brings us back to imaginary numbers, $-i$; and another brings us back to our original place, 1.

These rotations of ninety degrees, or one right angle, corresond to *exponentiation*. Figure 1↋ on page 3↊ demonstrates the powers of i and show the pattern which gives rise to the complex plane. In other words, as Figure 1↋ shows, the formula for a power of i is $i^n = i^{n \bmod 4}$. The patterns for exponentiation repeat every four powers.

We can further, as shown in Figure 1↊, isolate individual points on the

Power	Value	Power	Value	Power	Value
i^{-4}	1	i^0	1	i^4	1
i^{-3}	i	i^1	i	i^5	i
i^{-2}	-1	i^2	-1	i^6	-1
i^{-1}	$-i$	i^3	$-i$	i^7	$-i$
	0		0		0

Figure 1Ɛ: *Powers of i and their sums.*

complex number plane by combing a *real part* with an *imaginary part.* As suggested by the axes, the real part of such a point is its location along the x-axis, while the imaginary part is its location along the y-axis. The two are typically separated by a + or −, as the case may be. In our case, 0;6+0;9i reflects a point at 0;6 on the x-axis and 0;9 on the y-axis. The real part is typically labeled a, and the imaginary part b; so imaginary numbers in full form are typically written $a + bi$, though the a is usually left out if it is zero, and bi is also left out if b is zero. (These, containing only a, are in fact real numbers.) And so far all is simple and clear.

Just as on the Cartesian plane, altering the sign of a coordinate reflects it across the appropriate axis. 0;6+0;9i can be reflected across the real axis—the x-axis—by changing it to −0;6+0;9i; it can be reflected across the imaginary axis by changing it to 0;6−0;9i.

Multiplication by −1 *is a reversal of direction*; that is, rotating one hundred and eighty degrees around the origin. *Multiplication by i is a rotation of a single right angle*, or half a change in direction; that is, rotating ninety degrees around the origin. (Multiplying by $-i$ rotates ninety degrees in the other direction.) We can see this easily because $-i^2 = 1$ and $i = \sqrt{-1}$. As always, a full rotation around the circle on the complex plane changes nothing; we retain the same value.

There are entire sets of rules allowing calculation using complex numbers (though some operators we take for granted with real numbers, like comparisons, simply don't work for complex ones). The most basic is *conjugation*, the reflection of a complex number z aroundt he real axis. The answer will be represented by the symbol $\overline{z}$. But we've already seen how to do this: we simply make $z = a + bi$ and make it into $\overline{z} = a - bi$. This operation is important because a complex number is real *if and only if* it equals its own conjugate. Conjugate a complex number twice, and you will get the same number; once again we see the vital operation of *reflection across the axis*, which we've already seen as tending to favor a division into two dozen.

We've seen one way of encoding points onto the complex plane; now we examine another, the so-called *absolute value* of the point. This involves treating the point as an angle turn plus a distance to the point from the origin; this idea leads to the application of trigonometric functions to complex numbers. Given a point $a + bi$, the absolute value of that point is $|z| = \sqrt{a^2 + y^2}$; this also equals the distance of the point from the origin. For our point, the absolute value is:

$$|z| = \sqrt{0;6^2 + 0;9^2} = 0;↊997$$

To this absolute value we add an *argument*, which is written $\arg(z)$. The formula for this is actually quite complex, due to the trigonometry involved:

$$\arg(z) = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{if } a > 0 \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan\left(\frac{b}{a}\right) - \pi & \text{if } a < 0 \text{ and } b < 0 \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0 \\ \text{indeterminate} & \text{if } a = 0 \text{ and } b = 0 \end{cases}$$

Provided that $\arg(z)$ is expressed in radians (for which see Section 1.1 on page 5), this gives the angle, rotating counterclockwise from the x-axis (in other words, using $(1, 0)$ as our starting point, just as we did on the Cartesian plane), that we must turn before traversing the distance to our angle.

Why the tangent? Simply because we're using triangle-based trigonometry to determine the angle. Figure 20 on page 40 demonstrates what we mean.

If θ is the angle turned counterclockwise before beginning to travel from the origin to the point in question; and if d is the distance from the origin to the point in question (which we called earlier the point's absolute value); and if a is the side adjacent to θ; and if b is the side opposite θ; then we've constructed a right triangle and can do all the miraculous work of trigonmetry right here on the complex plane. (θ, of course, equals $\arg(z)$.)

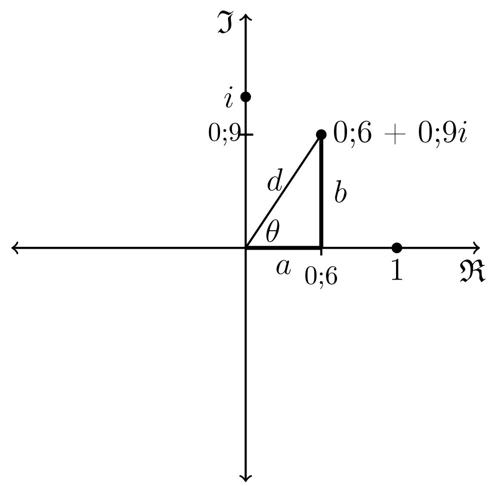


Figure 20: *Absolute value and argument location of a point in the complex plane.*

We know that the tangent of θ is $\frac{b}{a}$ (these correspond to the x- and y-coordinates, of course); so if we take the arctangent of this, we will get the value of θ in radians. In this case, $\arctan\left(\frac{0;9}{0;6}\right) = 0;↋963$ radians, or just over fifty-six degrees, which (unscientifically, of course) looks just about right from Figure 20. (This would be about 0;76 in four-dozen units, 0;576 in three-dozen units, 0;39 in two-dozen units, and 0;1↊ in one-dozen units.)

Notice the prevalence of π in our function $\arg(z)$ here. As we saw in Section 1.2 (starting on page 7), π (3;18480949...) is the number of radians in the half circle, implying a division of the circle into two dozen parts. Whenever we have a real part to the number (that is, whenver a complex number is not a *pure imaginary* number), the angle π radians (the number of radians in the half circle) is crucial; in fact, we transfer our complex numbers across the axes by adding or subtracting π radians.

Of course, whenever we have a pure imaginary number (one with no real part), $\frac{\pi}{2}$, the right angle, becomes more important; because, of course i is simply a rotation of ninety degrees.

Nowhere do we see the full circle; nowhere do we see a third of a circle.

However, we do see full circles in other areas of complex numbers, particularly in exponentiation of these numbers, wihch involves multiplying integers by 2π (ς) in order to find the nth root of complex numbers:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + k\varsigma}{n}\right) + i\sin\left(\frac{\theta + k\varsigma}{n}\right)\right)$$

But we will see more of these sort of thing in the next section.

At the end of this brief examination of imaginary and complex numbers, we see that reversals and π appear to play a cardinal role, and secondarily to that ninety-degree rotations are important. For this reason, the final score is 3, 1, 4, and 2.

4.2.2 Euler's Formula—3, 1, 4, 2

Euler's Identity is among the most famous formulas in mathematics, chiefly because it relates what are widely considered the five most important constants in mathematics—0, 1, e, π, and i—in a single, simple formula. Conventionally written $e^{i\pi} + 1 = 0$, it is in fact a special case of Euler's formula, which is

$$e^{ix} = \cos x + i \sin x$$

Euler's identity is just another way of writing $e^{i\pi} = -1$, of course; and since the cosine of π radians is -1, and the sine of π radians is 0, the entire second term cancels out if $x = \pi$, yielding the formula that generations of mathematicians have learned and loved.

But our other circle units also produce some quite interesting equations, with the exception of three dozen; this results in complex numbers (that is, non-pure imaginary numbers), and consequently will be left out of our discussion here. Figure 21 on page 41 shows Euler's identity of the unit of angle in the other three divisions.

One Dozen	Two Dozen	Four Dozen
$e^{i\varsigma} = 1$	$e^{i\pi} = -1$	$e^{i\eta} = i$

Figure 21: *Euler's identities arising from the complex plane according to the four divisions of the circle.*

Equalling unity is always a compelling result, so let's look a little more closely at Euler's identity in terms of ς, repeated here for closer inspection:

$$e^{i\varsigma} = 1$$

This is, indeed, very interesting. However, let's compare it to another equation:

$$e^{i0} = 1$$

Our first impression might be that this shouldn't work; but we must remember that the 0 cancels out the i, and any number to the power of 0 equals one, even e. But still, to be sure, let's plug it into Euler's formula, with $x = 0$:

$$e^{i0} = \cos 0 + i \sin 0 = 1 + i0 = 1 + 0 = 1$$

We're faced here once again with the fact that one full circle is an angle equivalent to *zero*; and so we often learn little by using it as a unit.

This does get more interesting to us as a unit, however, when we are studying roots of complex numbers; the sum of the roots of any complex number will equal zero, and thus the full circle unit (which, as an angle, is equivalent to zero). However, we've already seen by the nature of i and 1 that exponentiation (the opposite of roots) is equivalent to rotations around the circle, so it is unsurprising that $e^{i\varsigma/3} + e^{i2\varsigma/3} + e^{i3\varsigma/3} = 0$; we've already seen that that's the case with integer powers in Figure 1Ɛ on page 3Ƨ, so the fact that it works the same way with fractional powers (roots) isn't particularly enlightening.

On the other hand, we learn a great deal when we look at these identities in terms of the half-circle unit, or the division into two dozen, or π. We see that π is necessary for reversing the sign of any of these numbers, real or imaginary. This fact tells us more than ς does, because ς simply tells us that an angle is equal to itself; and it tells us more than η does, because η can only tell us about one increment in the exponent rather than two.

This reversal that occurs at π is also vitally important in the actual applications of complex numbers to real situations. For example, in electrical engineering, alternating current can be represented on a complex plane, with positive numbers being one direction of the current and negative numbers being the other. Engineers then take the real part of these calculations to determine what exactly they're dealing with.

We also see that π is the smallest number which will bring us back to the set of real numbers. This is important because while imaginary numbers are extremely useful in calculations, what we really want at the end of our calculations are measurable quantities, and that requires real numbers. Adding η to our quantities gives us some useful information; adding ς simply gives the same thing we started with; but adding π gives us what η gives plus more.

Therefore, Euler's formula yields scores of 3; 1; 4; and 2.

5 Trigonometry

Trigonometry is the study of triangles and the relationships within triangles of sides and angles. It is applicable to circles and to cyclical phenomena, like waves, alternating current, and the like, but fundamentally it is a *triangular* science, as its name itself implies.[†] It's worth noting along these lines that, although sine and cosine correspond to the y and x coordinates on a circle which will be thrown by a given angle, this only works in a *unit circle*; that is, one in which the radius is equal to 1. In other circles, one must make adjustments for the different radius. The trigonometric functions work in *every* triangle, however, no matter what the size. Given the importance of the half-circle angle to triangles, we can start to see the outcome of this category already.

Trigonometry has significant applicability for astronomy, surveying, construction, and many other fields.

First, we'll consider trigonometry in general, then some individual rules of it.

5.1 Trigonometry in General—2, 1, 4, 3

The mnemonic known to every student of trigonometry is "SOHCAHTOA": the *s*ine equals the side *o*pposite the angle divided by the *h*ypotenuse; the *c*osine equals the side *a*djacent to the angle divided by the *h*ypotenuse; and the *t*angent equals the side *o*pposite to the angle divided by the side *a*djacent to the angle. In other words:

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

[†]It is from $\tau\rho\iota\gamma o\nu o\nu$, meaning "triangle," and $\mu\epsilon\tau\rho o\nu$, meaning "measure."

$$\tan\theta = \frac{opposite}{adjacent}$$

These equations lead to the additional equation:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

There are also *reciprocal* trigonometric functions, which are known as the cosecant (reciprocal to sine); secant (reciprocal to cosine); and cotangent (reciprocal to tangent):

$$\csc\theta = \frac{1}{\sin\theta} = \frac{hypotenuse}{opposite}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{hypotenuse}{adjacent}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{opposite}{adjacent} = \frac{\cos\theta}{\sin\theta}$$

There are also *inverse* trigonometric functions, the arcsine, arccosine, arctangent, and so on. These are a bit more complicated; but essentially they are the opposite of the trigonometric functions:

$$(y = \arcsin x) = (x = \sin y)$$

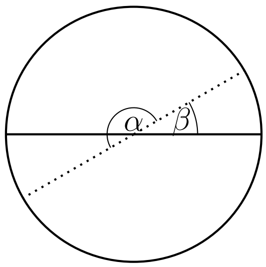
$$(y = \arccos x) = (x = \cos y)$$

$$(y = \arctan x) = (x = \tan y)$$

There are similar inverse functions for the reciprocal functions, as well. As the equations demonstrate, these functions are effectively inserting the trigonometric function value and getting out the correct angle, rather than the other way around. For example, $\sin 15$ (for the sine of fifteen degrees) equals 0.2588; $\arcsin 0.2588 =$ fifteen degrees.

Trigonometric functions can be extended past right angles by using the *unit circle*, that circle whose radius is 1. When we do this, we see that these functions are *cyclical*; that is, they repeat at certain values. It's easy enough to graph these functions, so we won't do that here; we will, however, examine some of the facts about these functions in light of such graphs:

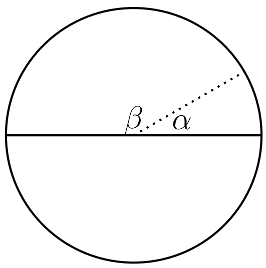
1. Sine, cosine, cosecant, and secant have periods equal to the full circle; that is, they repeat themselves after a full circle's worth of angles.



Opposite Angles

Opposite angles are those which differ by exactly one straight line.

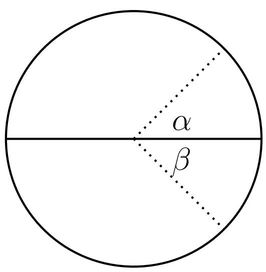
1. All functions have identical absolute values.
2. Sine, cosine, secant, and cosecant have opposite signs.
3. Tangent and cotangent have the same signs.



Supplementary Angles

Supplementary angles are those which, when added together, equal a straight line.

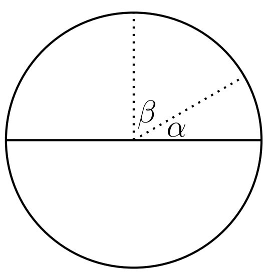
1. All functions have identical absolute values.
2. Sine and cosecant have the same sign.
3. Cosine, tangent, secant, and cotangent have opposite signs.



Explementary Angles

Explementary angles are those which are equal on either side of the circle's diameter; or alternatively those which together equal a full circle.

1. All functions have identical absolute values.
2. Cosine and secant have the same sign.
3. Sine, cosecant, tangent, and cotangent have opposite signs.



Complementary Angles

Complementary angles are those which, when added together, make up a single right angle.

1. Sine and cosine have identical absolute values for complementary angles.
2. They have the same sign for angles less than ninety degree and for angles between one hundred eighty and two hundred seventy degrees.
3. They have opposite signs for other angles.

Figure 22: *Various types of angles mapped to their trigonometric properties.*

2. Tangent and cotangent have periods equal to half a circle; that is, they repeat themselves after a half circle's worth of angles.
3. *All the trigonometric functions* have identical absolute values for *opposite angles*; that is, for those angles which differ by precisely one straight line, or one half a circle, or two right angles. For sine and cosine, opposite angles will have opposite signs; for tangent, opposite angles will have the same sign.
4. *All the trigonometric functions* have identical absolute values for *explementary angles*; that is, for an angle on one side of the circle's diameter, the identical angle on the opposite side of the circle's diameter will have the same absolute value. Having the same *absolute value* means that, if the negative sign (if any) is removed, the numbers will be the same. We can analyze these angles as *negative angles* (simply negating the angular measure) or as adding up to a full circle. This is true both for those functions with a period of a full circle and those functions with a period of a half circle.
5. *All the trigonometric functions* have identical absolute values for *supplementary* angles; that is, for those angles whose sum is a half circle. This is true both for those functions with a period of a full circle and those functions with a period of a half circle.
6. Sine and cosine equal one another for *complementary* angles; that is, for those angles whose sum is a right angle. They have the same absolute value no matter what; they have identical values if the angle is between zero and ninety degrees or between one hundred eighty and two hundred seventy degrees.

Looking at Figure 22 on page 45, it's impossible not to notice two things: 1. the full circle is vitally important; 2. the half circle is vitally important. It's also impossible not to notice that the *half-circle* appears to be even more important than the whole. It certainly appears more often, and the helpful equivalencies of the trigonometric functions appears most often to center around the half-circle.

It could be argued that the period of sine and cosine, being equal to a full circle, makes the full circle more important than the half; however, that ignores the period of the tangent being a half circle, as well as the many congruities between angles which are based on the half circle. Supplementary angles and opposite angles are the most obvious; but even explementary angles clearly reflect a single angle reflected across a straight line, which is itself the angle of a half circle.

It is clear, then, that this category is a victory for the division of the circle into half. The remaining divisions are more difficult to grade; specifically, we find ourselves with a tough competition between one and four.

Four has the benefit of being a right angle; not only do two right angles make up a straight line, which we've already seen is so important, but the right triangle is what makes the whole science of trigonometry possible; that is, without the right triangle trigonometry may never have been discovered. On the other hand, the full circle clearly seems to have a close relationship to these questions, particularly given that the period of two of three trigonometric functions is a full circle. All told, one dozen should take precedence over four dozen. Three dozen takes fourth.

5.2 Trigonometry and the Unit Circle

We described trigonometry as arising out of the right triangle, and indeed it does; however, it extends to arbitrary angles in part due to its relationship to the *unit circle*. In each quadrant of the unit circle, we can inscribe an infinite number of right triangles, with the right angle at the origin. Doing this, we can relate all the functions to the unit circle, and apply trigonometric functions to arbitrary angles.

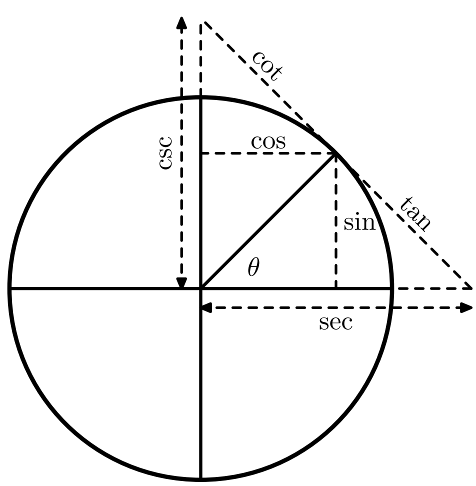


Figure 23: *Trigonometric functions inscribed in the unit circle.*

Figure 23 on page 47 shows how these trigonometric functions can be mapped onto a unit circle. Fundmentally, these trigonometric values are *ra-*

tios; and as such, they are simply numbers, dimensionless quantities. However, when we imagine them as figures in a unit circle as depicted in Figure 23, we can make them into *lines*; and this allows some interesting calculations to be performed.

For one thing, as already noted, this makes doing trigonometric functions on arbitrary angles, even those greater than a right angle, easy, just as easy as the acute angles that SOHCAHTOA had already made so simple. However, to do this we must remember that *sine, cosine, tangent, and cotangent are negative if their direction is opposite to what it is in the first quadrant*; while *secant and cosecant are negative if their direction is opposite to the radius*.

At first glance, this appears a great victory for the division of the circle into one dozen parts; for it is the use of the full circle that extends the trigonometric functions to all angles. However, when we remember this requirement regarding signs (positive and negative), we are returned again to the division into two dozen; or, indeed, four dozen parts, because the patterns that these signs follow will be according to those two divisions. Furthermore, what we've essentially done in Figure 23 is inscribe a number of right triangles into a quadrant of the circle, and remembering that whenever we map these functions onto a circle we're mapping right triangles onto it, with all the conclusions that we drew regarding triangles in our sections on geometry.

Figure 27 on page 54, in Section 5.5, examines these patterns, and they are graded separately there, so there is no grade for this subsection. This is merely to give some background to how these patterns develop.

5.3 Formal Definitions

For the purposes of further reviewing the trigonometric functions and which division of the circle makes them most transparent and easy to use, we will embark now upon a formal definition of those functions in terms of their *domains*, their *ranges*, and their *periods*.

Without getting too technical, the *domain* of a function is the set of valid inputs to it; the *range* of the function is the set of valid outputs; and the *period* is the total cycle of the function (when it begins repeating). These definitions are curt and imprecise, but they will do for our purposes.

Note that the inverse functions (those with "arc" at the front of their names) have no period.

As usual, let c equal the unit of the full circle:

Function	Domain	Range	Period
$\sin\theta$	$\mathbb{R}$	$[-1, 1]$	c
$\cos\theta$	$\mathbb{R}$	$[-1, 1]$	c
$\tan\theta$	$\theta \neq \left(n + \frac{1}{2}\right)\left(\frac{c}{2}\right), n \in \mathbb{Z}$	$\mathbb{R}$	$\left(\frac{c}{2}\right)$
$\sec\theta$	$\theta \neq \left(n + \frac{1}{2}\right)\left(\frac{c}{2}\right), n \in \mathbb{Z}$	$(-\infty, -1]; [1, \infty)$	c
$\csc\theta$	$\theta \neq n\left(\frac{c}{2}\right), n \in \mathbb{Z}$	$(-\infty, -1]; [1, \infty)$	c
$\cot\theta$	$\theta \neq n\left(\frac{c}{2}\right), n \in \mathbb{Z}$	$\mathbb{R}$	$\left(\frac{c}{2}\right)$
$\arcsin n$	$[-1, 1]$	$[-\frac{c}{4}, \frac{c}{4}]$	
$\arccos n$	$[-1, 1]$	$[0, \frac{c}{2}]$	
$\arctan n$	$\mathbb{R}$	$(-\frac{c}{4}, \frac{c}{4})$	
$\text{arccsc}\, n$	$(-\infty, -1]; [1, \infty)$	$[-\frac{c}{4}, 0); (0, \frac{c}{4}]$	
$\text{arcsec}\, n$	$(-\infty, -1]; [1, \infty)$	$[0, \frac{c}{4}); (\frac{c}{4}, \frac{c}{2}]$	
$\text{arccot}\, n$	$\mathbb{R}$	$(-\frac{c}{4}, 0); (0, \frac{c}{4}]$	

Figure 24: *Details of the formal definitions of the trigonometric functions.*

The symbol $\mathbb{R}$ is called a "blackboard bold," and represents the set of all real numbers; the symbol $\mathbb{Z}$ represents the set of all integers; the symbol $\in$ indicates that the preceding symbol belongs to the following set. The parenthesis, or "paren," indicates an *open interval*; that is, one which does not include its endpoint. The square bracket indicates a *closed interval*, one which does include its endpoint. A *mixed interval* has both of these; e.g., $[-\frac{c}{4}, 0)$ indicates an interval which *does* include $\frac{c}{4}$, but does *not* include 0.

There will be no score in this category; this chart serves simply to provide us information for future tests.

5.4 Laws of Trigonometry

Trigonometry has a number of *laws* which give us a great deal of assistance in making calculations. We will examine these one by one. For the purposes of these laws, we will use the triangle depicted in Figure 25 on page 47.

5.4.1 Law of Sines—2, 1, 4, 3

The Law of Sines (which applies to any arbitrary triangle) relates the sines of the angles and the lengths of the sides of that triangle to the radius of a

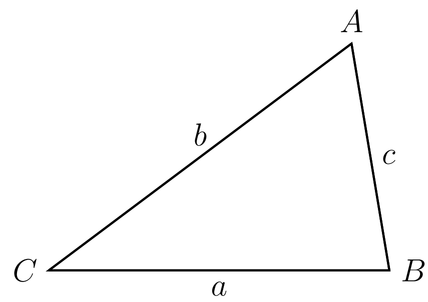


Figure 25: *Triangle to be used for trigonometric explorations.*

circumscribed circle; or rather, to the diameter of the circumscribed circle. Let r be the radius of that circumscribed circle; then:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

Curiously, these things are equal to $2r$, or the diameter; which is extra curious due to the formula for determining π, given C as the circumference of the circle and r as its radius:

$$\pi = \frac{C}{2r}$$

But π is the number of radians in the half-circle. Combining these two equations, and remembering our symbols for the number of radians in the various divisions of the circle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{2circ}{\varsigma} = \frac{circ}{\pi} = \frac{2circ}{3\psi} = \frac{circ}{2\eta}$$

The simplest of these identities is plain: the Law of Sines relates best to the number of radians in a half circle.

That radius can also be calculated from the sides of the triangle:

$$r = \frac{abc}{\sqrt{(a+b+c)(a-b+c)(a+b-c)(b+c-a)}}$$

This allows us to determine the angles by plugging that radius into the Law of Sines. For example:

$$2r = \frac{a}{\sin A}$$
$$2r \sin A = a$$
$$\sin A = \frac{a}{2r}$$

We can then use the arcsin to determine what the angle A is from its value $\sin A$.

It also means that, given two sides and the angle between them, we can calculate the area of the triangle, even without knowing its height. Let A equal the area of the triangle; then:

$$A = \frac{1}{2} ab \sin C$$

The Law of Sines, when considered as a whole, has little in it to favor any of our divisions. However, given its equality to twice the radius of a circumscribed circle, and that value's vital role in producing the number of radians in a half circle; and given its usefulness with triangles, the sum of whose interior angles is equivalent to a half circle; the division into two dozen takes first place. After that, the one dozen takes second, because by dividing the equality of the Law of Sines in two, we can produce the radius, which is what produces the full circle.

There appears to be no other reason to favor three or four dozen to the other; however, since division by two is easier than division by three, we will grade them accordingly.

5.4.2 Law of Cosines

We're all familiar with the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

However, we're also all familiar with the fact that this only works with right triangles. Trigonometry—specifically, the Law of Cosines—allows us to extend that rule to *any* triangle:

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$c^2 + 2ab \cos C = a^2 + b^2$$

$$2ab \cos C = a^2 + b^2 - c^2$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

There seems no reason to favor any division of the circle from this; therefore, it is not graded.

5.4.3 Law of Tangents

The Law of Tangents relates angles with their opposite sides:

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{1}{2}(A-B)\right)}{\tan\left(\frac{1}{2}(A+B)\right)}$$

As with the Law of Cosines, there seems no reason to favor any division of the circle from this; therefore, it is not graded.

5.5 Patterns in Function Values—3, 1, 4, 2

As we did when considering simply the measurements of some common angles, it behooves us to consider also the trigonometric values of some of these common angles, and observe where they take us. Figure 26 on page 51 gives some such angles.

As we can see in that Figure, these angles repeat for the sine and cosine every three hundred and sixty degrees (one full circle), and for the tangent every one hundred and eighty degrees (one half circle). However, we see definite patterns that go deeper than this superficial examination.

First off, the right angle is plainly very important for these functions; and this makes sense, given that the whole system was developed for dealing with right triangles, and is in fact used most commonly (in navigation, surveying, and such tasks) in reference to right triangles. The right angle shows up in the following ways:

1. At right angles (ninety, one hundred and eighty, and two hundred and seventy degrees), the sine and cosine are simple integers, and the tangent is either undefined (at ninety and two hundred and seventy), or a simple integer.
2. At half the right angles (forty-five degrees, one hundred thirty-five degrees, two hundred twenty-five degrees, and three hundred and fifteen

Deg. (Dec.)	Sine	Cosine	Tangent
0	0	1	0
5.625	**0;1214**	**0;↋↋38**	**0;1222**
11.25	**0;2411**	**0;↋929**	**0;2478**
15	0;3132	0;↋711	0;3270
22.5	**0;4713**	**0;↋105**	**0;4↋79**
30	0;6	0;↊485	0;6↋17
40	*0;7868*	*0;9238*	*0;↊09↋*
45	**0;859↊**	**0;859↊**	**1**
60	0;↊485	0;6	1;894↋
75	0;↋711	0;3132	3;894↋
80	*0;↋998*	*0;2100*	*5;807↋*
90	**1**	**0**	**undef.**
105	0;↋711	-0;3132	-3;894↋
120	*0;↊485*	*-0;6*	*-1;894↋*
135	0;859↊	-0;859↊	-1
150	0;6	-0;↊485	-0;6↋17
165	0;3132	-0;↋711	-0;3270
180	**0**	**-1**	**0**
195	-0;3132	-0;↋711	0;3270
210	-0;6	-0;↊485	0;6↋17
225	-0;859↊	-0;859↊	1
240	*-0;↊485*	*-0;6*	*1;894↋*
255	-0;↋711	-0;3132	3;894↋
270	**-1**	**0**	**undef.**
285	-0;↋711	0;3132	-3;894↋
300	-0;↊485	0;6	-1;894↋
315	-0;859↊	0;859↊	-1
330	-0;6	0;↊485	-0;6↋17
345	-0;3132	0;↋711	-0;3270
360	0	1	0

Figure 26: *Trigonometric values of common angles.*

degrees), the sine and cosine have identical absolute values. At forty-five degrees and two hundred twenty-five degrees, the sine and cosine have identical values, including sign.

3. The above is a special case of this rule: *complementary* angles have identical sines and cosines. Complementary angles are those which, when added, make up a right angle. So, for example, the sine of ten degrees equals the cosine of eighty degrees; the sine of sixty degrees equals the cosine of thirty degrees. The sines and cosines might differ in sign, but their absolute value is always the same.
4. The tangents of angles climb to infinity at the right angle, then negate, then climb back to zero at two right angles, and repeat the process from two to four right angles.
5. While the entire pattern for sine and cosine repeats only after a full circle, the subsidiary patterns are clearly based on the right angle; and while the entire pattern for tangent repeats after a half circle, the subsidiary pattern is also clearly based on the right angle.
6. As seen in Figure 24 on page 49, the domains for the tangent and the secant both involve $\left(n+\frac{1}{2}\right)\left(\frac{c}{2}\right)$, which of course is equal to $n\left(\frac{c}{2}\right)+\frac{\left(\frac{c}{2}\right)}{2}$, which itself is equal to $n\left(\frac{c}{2}\right)+\left(\frac{c}{4}\right)$. But $\frac{c}{4}$ is the number of radians in a right angle; so the domains of these two functions is dependent upon the right angle.
7. The inverse functions appear heavily dependent upon the right angle. Arcsine, arcsecants, arccosecants, arctangents, and arccotangents all have ranges which depend heavily on $\frac{c}{4}$. This is due to the angle patterns we see here; because the patterns often repeat at $\frac{c}{4}$, the angles have the same function values, so for those values the inverse functions can only return those less than $\frac{c}{4}$.

All told, this appears to be a resounding victory for dividing the circle into four dozen. However, the division into two dozen still carries some important advantages over four:

1. The half-circle is twice the right angle; this captures something of the trigonometric patterns which the right angle enjoys.
2. Some of these right angle trigonometric patterns involve reversals when passing the right angle; e.g., the patterns in the tangent functions. The right angle misses these broader patterns, while two right angles captures them. These two points are illustrated graphically in Figure 27 on page 54.

3. Additional patterns in the trigonometric functions appear, including those reviewed in Figure 22 on page 45.
4. As seen in Figure 24 on page 49, the shortest period for a trigonometric function is $\left(\frac{c}{2}\right)$ radians; or, the number of radians in a half-circle. This means that the smallest unit which can nonfractionally express the periods of all these functions is the half-circle, which supports the half-circle (and thus the division of the circle into two dozen) as the unit of angle.
5. As seen in Figure 24 on page 49, the only functions which can result in any real number are those with a period of a half-circle. This means that the period of the half-circle entails the greatest range of numbers (all real numbers), which favors the division of the circle into two dozen.
6. As seen in Figure 24 on page 49, those functions with limited domains have domains which depend upon $\left(\frac{c}{2}\right)$ (the half circle) or multiples thereof. It is most important to know the domain, because that tells us the set of valid inputs to a function. This favors the half-circle, and thus the division of the circle into two dozen.
7. As seen in Figure 24 on page 49, the range of the arccosine is dependent upon $\frac{c}{2}$, and the upper bound of the arcsecant is $\frac{c}{2}$.

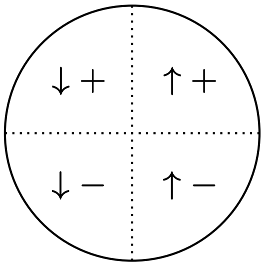
The full circle carries some benefits here, of course; the periods of sine and cosine are equal to the number of radians in a full circle, for example. But this isn't enough to override the benefits of two dozen and four dozen. Having only one dozen units in the entire circle obscures these relationships, making it more difficult to see the patterns in the trigonometric function values. Rather than adding up to one for supplementary angles, or one half for complementary, one must add up to half for supplementary angles, or one quarter for complementary. The relationship between 0;26 and 0;06 is not as facially clear as that between 0;1 and 0;5 (for complementary angles), for example. The division into one dozen makes even these fundamental divisions between trigonometric patterns fractional, which makes them harder to work with.

As usual, three dozen has little to recommend it here, giving it last place.

5.6 Trigonometric Identities—3, 1;6, 4, 1;6

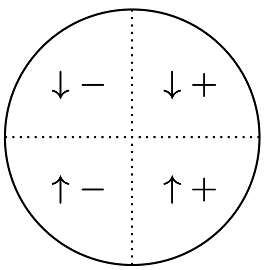
Trigonometric identities are equations which hold true for all angles, regardless of size. These identities will work regardless of which division of the circle we select; however, they may be made more transparent by one such

Sine



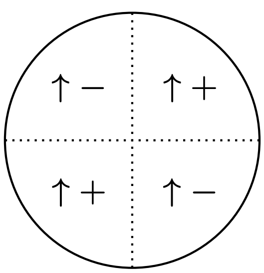
1. In the first quadrant, the values are positive and increasing with angle.
2. In the second quadrant, the values are positive and decreasing as angle increases.
3. In the third quadrant, the values are negative and decreasing as angle increases.
4. In the fourth quadrant, the values are negative and increasing with angle.

Cosine



1. In the first quadrant, the values are positive and decreasing while angle increases.
2. In the second quadrant, the values are negative and decreasing while angle increases.
3. In the third quadrant, the values are negative and increasing with angle.
4. In the fourth quadrant, the values are positive and increasing with angle.

Tangent



1. In the first quadrant, the values are positive and increasing with angle.
2. In the second quadrant, the values are negative and increasing with angle.
3. In the third quadrant, the values are positive and increasing with angle.
4. In the fourth quadrant, the values are negative and increasing with angle.

Figure 27: *Pictorial representation of the patterns in trigonometric functions.*

division than by another. Determining whether this is the case is our goal here.

There are three of significant importance, which we will review one at a time.

$$\sin^2 A + \cos^2 A = 1$$

Take note that when the exponent is attached to the function, then one applies it to the *result* of the function; when it is attached to the function's argument (in this case A), one applies it to the argument. So here we want "$(\sin A)^2$," not "$\sin A^2$."

This curious fact is true regardless of which angles we select; but it more obviously correct (more obviously to the human eye, that is, which works better with simple numbers than with complex ones) if we select certain angles. Specifically, if we select *right angles.* If we select the first right angle, for example, we get $1^2 + 0^2 = 1$, which is trivial; if we select the second, we get $0^2 + -1^2 = 1$, which is only slightly less trivial; and similarly for the third or fourth. This favors the right angle as our division of the circles; that is, division of the circle into four dozen.

$$\sec^2 A - \tan^2 A = 1$$

Once again, this curiosity is true regardless of angle; but it's more obviously correct with certain angles. (Those angles where the tangent is undefined are obviously excepted.) The secant is the reciprocal of the cosine, so we should look, as last time, for those angles where the cosine would yield a simple reciprocal. This gives us most obviously those angles where the cosine is equal to a simple integer; we must also except those angles where the cosine is equal to 0, because we cannot take the reciprocal of zero. (The secant at these angles is undefined.) That means that we're left with the cosine at zero degrees; one hundred eighty degrees; and three hundred sixty degrees. In other words, we're left with angles at one hundred and eighty degree increments.

Simple fractions also yield relatively simple reciprocals; e.g., the reciprocal of 0;6 is 2. These will give us *simpler* identity equations of this type, though the tangent squared is still a difficult number. But these come at sixty degree intervals; that is, a third of a straight line. These are thus absorbed by dividing the circle into two dozen, as well.

$$\csc^2 A - \cot^2 A = 1$$

The cosecant is the reciprocal of the sine, and the cotangent the reciprocal of the tangent; so if we discount those angles where the two are undefined (because the primary functions either equal zero or are themselves undefined), we see that there really are no easy angles to see this in, at least not in any helpful pattern.

Another set of interesting trigonometric identities can be seen in Figure 28 on page 56. Really, of course, there is nothing new here; we have seen all of this in Figure 27 on page 54. However, it is still interesting to see this laid out this way, as it helps us determine the patterns behind these identities.

$$\begin{aligned}
\sin\left(\theta + \tfrac{c}{4}\right) &= \cos\theta \\
\sin\left(\theta + \tfrac{c}{2}\right) &= -\sin\theta \\
\sin\left(\theta + \tfrac{3c}{2}\right) &= -\cos\theta \\
\sin\left(\theta + c\right) &= \sin\theta
\end{aligned}$$

Figure 28: *Some important trigonometric identities.*

Fundamentally, of course, the right angle, $\frac{c}{4}$, is plainly the basic unit behind all of this; and that makes sense, as the basis for all trigonometry is fundamentally the right angle. This fact favors the division of the circle into four dozen parts. And interestingly, we can see that the pattern comes full circle (no pun intended) at the unit of the full circle, c, which favors the division of the circle into one dozen parts. However, both of these divisions miss the essential part of the pattern: that the identical numerical value (that is, the identical absolute value) comes about at intervals of one straight angle, $\frac{c}{2}$, the sign reversing itself at each straight angle. This pattern captures both that of $\frac{c}{4}$ *and* that of c, a strong endorsement for division of the circle into two dozen parts.

Another way of viewing this is to chart all of these trigonometric identities in terms of *symmetry*, a chart which will be another way of looking at what we've already observed in Figure 27 on page 54. We can see these patterns in Figure 29 on page 57. In that Figure, f indicates the trigonometric function indicated in the left column, and θ is an arbitrary angle.

Figure 29 is essentially complete; while we could add a column for $f(2\pi - \theta)$, this would merely duplicate one of the columns we've already produced

	$f(-\theta)$	$f(\frac{c}{4}-\theta)$	$f(\frac{c}{2}-\theta)$	$f(\frac{3c}{2}-\theta)$
sin	$-\sin\theta$	$\cos\theta$	$\sin\theta$	$-\cos\theta$
cos	$\cos\theta$	$\sin\theta$	$-\cos\theta$	$-\sin\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\tan\theta$	$\cot\theta$
csc	$-\csc\theta$	$\sec\theta$	$\csc\theta$	$-\sec\theta$
sec	$\sec\theta$	$\csc\theta$	$-\sec\theta$	$-\csc\theta$
cot	$-\cot\theta$	$\tan\theta$	$-\cot\theta$	$\tan\theta$

Figure 29: *Trigonometric symmetry charted.*

(that is, it would duplicate $f(-\theta)$). Indeed, arguably $f(-\theta)$ and $f(2\pi-\theta)$ are redundant; they describe the identical angle.

(Furthermore, saying $\sin(\theta+c)=\sin\theta$ is also arguably redundant, as $\theta+c$ and θ are exactly the same angle, at least when considered as parts of a circle.)

Overall, the trigonometric identities seem to about evenly favor the division into two and four dozen. The division into three dozen doesn't expose any easy patterns for these, while the division into one dozen is simply too large, missing out on the smallest units of these patterns.

5.7 Trigonometric Mnemonics—3, 2, 4, 1

In the days of calculators, we often forget about how important and convenient it can be to remember certain very important results of mathematical functions. The most important trigonometric angles, though, follow a pretty simple pattern; so we will examine this and see which, if any, of ours circular divisions it favors.

Figure 2Ɛ on page 58 shows, separated by semicolons, the units concerned, where u indicates, as always, the basic circle unit. The chart obviously varies from an angle of 0 to an angle of thirty degrees, and comprises the angles zero degrees; thirty degrees; forty-five degrees; sixty degrees; and ninety degrees.

For one dozen, the divisions proceed in increments from zero, to 0;1, to 0;16, to 0;2, to 0;3. For two dozen, they proceed from zero, to 0;2, to 0;3, to 0;4, to 0;6. For three dozen, they proceed from zero, to 0;09, to 0;46, to 0;6, to 0;9. For four dozen, they proceed from 0, to 0;4, to 0;6, to 0;8, to 1.

Obviously, four dozen here is the easiest to deal with; it proceeds quite

Sine	Cosine	Value
0;0;0;0	$\frac{u}{4};\frac{u}{2};\frac{3u}{4};u$	$\sqrt{0}/2$
$\frac{u}{10};\frac{u}{6};\frac{u}{14};\frac{u}{3}$	$\frac{u}{6};\frac{u}{3};\frac{u}{2};\frac{2u}{3}$	$\sqrt{1}/2$
$\frac{u}{8};\frac{u}{4};\frac{3u}{8};\frac{u}{2}$	$\frac{c}{8};\frac{u}{4};\frac{3u}{8};\frac{u}{2}$	$\sqrt{2}/2$
$\frac{u}{6};\frac{u}{3};\frac{u}{2};\frac{2u}{3}$	$\frac{u}{10};\frac{u}{6};\frac{u}{14};\frac{u}{3}$	$\sqrt{3}/2$
$\frac{u}{4};\frac{u}{2};\frac{3u}{4};u$	0;0;0;0	$\sqrt{4}/2$

Figure 2Ƨ: *Mnemonic for the memorization of certain important angles and their trigonometric values.*

simply according to simple uncias. The vulgar fractions are sometimes slightly gnarlier than other divisions (e.g., $\frac{2u}{3}$), but not badly so.

After the four dozen, the two dozen is the simplest. Proceeding as it does from 0;2, to 0;3, to 0;4, to 0;6, all the important fractions of the dozen are covered (the sixth, the quarter, the third, and the half). Furthermore, the vulgar fractions are extremely simple.

The one dozen comes next. The uncias of this division are too large; the produce even inline fractions that are a bit unwieldy at times (0;16), and the vulgar fractions, though simple, are large. It does, though, take up the sixth, the quarter, and the uncia, as well as the eighth; these are important fractions, even if not as important as the third and the half.

The three dozen is, plainly, gnarly and inconvenient, and must come in last place.

6 Uses of Angle

We've discussed many of the uses of angle in passing; however, in this section we select some of the most important and common uses of angle, and discuss how angles are handled. If angles are handled in a certain way across many different practices, this is an *indication*, though not dispositive, that this is the easier way to handle them. Also, it is an advantage for a new metric system which handles angles in the same way, for it minimizes the upheaval that conversion to that new system would require.

6.1 Time and the Calendar—2, 1, 4, 3

For working with time, we here on Earth are faced with certain necessary cycles: the day and night cycle, the lunar cycle, the solar cycle. We've chosen to make a combination of the lunar cycle (months) and the solar cycle (years) our chief measure of the passage of time. Of our months, four have 26 days, five have 27, and one has only 24, or 25 in leap years. Our year, of course, has 265 days, or 266 in a leap year.

We divide the shorter cycles of time into *days*, which themselves are divided into a light period and a dark period. Historically, each of these periods was divided into twelve (10) parts; however, for a long time we have simply divided the entirety of the two periods into two dozen parts regardless of when light and dark occur. Those two dozen parts are called *hours*.

These are numbers that cannot really be avoided. (It is worth noting, however, that some calendars, notably the Symm010 calendar, can significantly regularize our month lengths, leaving eight months with 26 days and four with 27.) Any system that wants to deal with time has to adapt to these periods. Given that our view of time works as a cycle, and that historically we have viewed time as mapped onto a circle, this relates to angular measure in that it would be nice, if possible, to have our angular measure correspond to our divisions of the primary units of time.

Depending on which unit we select, we will have different numbers to deal with. Assuming angular measure based on a full circle unit, for example, we have twelve two-hour-long "duors," while assuming angular measure based on the half-circle we have two dozen one-hour-long hours. There will be different numbers of these hours in the important longer time periods that we must inevitably deal with; Figure 2Ɛ on page 5Ƨ gives a few of the most important of these figures.

It seems clear that the "roundest" figures, in the sense of having the most zeroes, are found in the division of the day into two dozen hours. This division has the added benefit of being a "least change" proposition: it's precisely the way we already do things, even in a decimal world.

On the other hand, division into one dozen gives some interesting advantages; for example, the number of hours in a year is simply the number of days in the year multiplied by the dozen.

However, in terms of simple facility in normal arithmetic, the numbers in the column of the two dozen are superior, sufficiently so to easily outweigh one dozen's advantage of not requiring a multiplication by two. Take the

Division	Week	26-day Month	27-day Month	Year	Leap Year
One Dozen	70	260	270	2650	2660
Two Dozen	120	500	520	50↊0	5100
Three Dozen	190	760	790	7730	7760
Four Dozen	240	↊00	↊40	↊180	↊200

Figure 2↋: *Units of time and associated numerals of the several divisions of the circle.*

number of time units in a week, for example. At first glance, one dozen's is eminently simple: seven days in the week, times 10 hours in a day, equals 70. However, 70 is divisible only by 2, 3, 4, 6, 7, 10, 12, 19, 24, and 36; while 120 is divisible by the same numbers, plus 8, 20, 28, 48, and 70 itself. Figure 30 on page 5↊ shows the relative divisibility of the numbers on hand in the time system.

Unit	One Dozen	Two Dozen	Three Dozen	Four Dozen
Week	↊	12	14	16
26-day Month	1↊	24	26	2↊
27-day Month	↊	12	14	16
Year	1↊	26	2↊	32
Leap Year	1↊	24	26	2↊

Figure 30: *Number of factors of time units according to various divisions of the circle.*

At first glance, this chart is entirely unsurprisingly; naturally, larger numbers will often have more factors than smaller numbers, and thus be easier to work into calculations. However, there is more here than meets the eye. The *rate of increase* is, of course, much higher for some of these divisions of the circle than for others. Figure 31 on page 5↋ demonstrates the different increases in numbers of factors for the important time units we're considering, with "rate of increase" very loosely defined as simply the average increase per unit of time.

Plainly, there is a very sharp jump in the average increase of factors

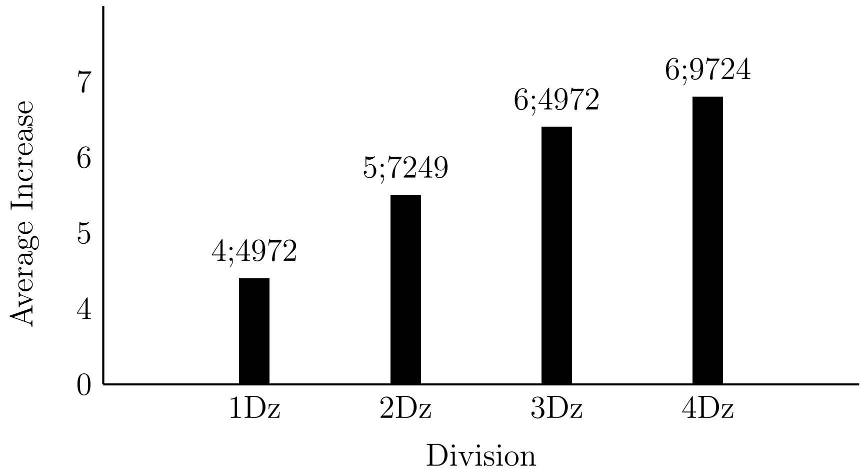


Figure 31: *Average increase in number of factors for time units in the various divisions of the circle.*

from one dozen to two dozen (of 1;2496), followed by a much less sharp increase from two dozen to three dozen (of 0;9725), followed by a still less sharp increase to four dozen (of 0;4971). We get, by a significant margin of approximately one quarter, the greatest increase of factors by shifting from a time unit based on one dozen units in a day to a time unit based on two dozen units in a day, with diminishing returns on further divisions thereafter, at least up to four dozen.

Furthermore, in combination with astronomical units (for which see Section 6.2, beginning on page 5Ɛ), the division into two dozen is powerful in the field of time measurements indeed.

6.2 Astronomy

Astronomers use angle constantly, and often in ways that do not really touch upon our discussion here; or at least which touch upon it in ways that are beyond the experience of the common user of a metric system, such as solid angle and similar things. We will therefore limit ourselves to such considerations of angle as an amateur astronomer is likely to use, which can itself get more complex than most people are familiar with.

Astronomers view the sky as what they call *the celestial sphere*; essentially, this imagines that all stars and planets are simply dots on a great glass dome which surrounds the entire Earth. There is a special line, called the *ecliptic*, which is the path that the sun takes on the celestial sphere. However, Earth is tilted on its axis; this means that the celestial sphere has an equator which is different from the ecliptic. All the planets move along the ecliptic, and appear very close thereto.

The axis of rotation on the celestial sphere is an imaginary line which proceeds out of both the north and south poles of Earth and proceeds all the way to the celestial sphere. Thus, the celestial sphere has *poles*, just as Earth does.

The star Polaris, often called the North Star, is situated almost identically on the north pole of the celestial sphere, around which all the northern hemisphere of the celestial sphere rotates. The celestial sphere's rotation is, of course, only apparent, caused by Earth's own rotation; this gives rise to *sidereal time*, from the Latin *sidus*, meaning "star," which is different from the normal time, which we judge at least nominally by the sun.

The *sidereal day* measures the rotation of Earth relative to the fixed stars; the *solar day* measures that rotation relative to the Sun. Because the Earth is also revolving around the sun, but not around the fixed stars, the sidereal day is slightly shorter than the solar day, by nearly four minutes; more precisely, by about 9;534370 biquaTims.

6.2.1 Altitude and Azimuth—3, 1;6, 4, 1;6

While right ascension and declination are used to identify an object's location on the celestial sphere,[‡] altitude and azimuth, known colloquially as "alt-az," identify an object's location in the sky at a given time.

Azimuth is a measure of the object's horizontal location around the horizon relative to the viewer. The viewer uses due north as a zero point and turns to the right (toward the east), counting upward in the degrees to which we are all accustomed. He counts up to three hundred and fifty-nine degrees, then returns to zero. If the azimuth of the object is *less* than one hundred eighty degrees, it is *rising*; if it is *more* than one hundred eighty degrees, it is *setting*. Therefore, there is an important primary division of the circle into two parts, a rising part and a setting part.

This practice clearly supports the division of the circle into two, and only then into subsidiary parts. That, in turn, supports dividing the circle into

[‡] *See supra*, Section 6.2.2, at page 61.

two dozen.

Altitude, obviously, is the height above the horizon. This is envisioned as a point along a glass shell which encases the planet from horizon to horizon. This is a unit from zero to ninety degrees; this works fine because the azimuth is given first. This is clearly an indication of the division of the circle into four dozen parts; though only two of those four dozen are visible.

So this is a mongrel two-dozen and four-dozen system; so we'll give them equal points.

6.2.2 Right Ascension and Declination—3, 1;6, 4, 1;6

We also have a system known as *right ascension* and *declination*, which are used to identify an object's location on the celestial sphere; this is an *objective* location, as objects will have the same right ascension and declination at all times. This is an objective system, rather than a subjective one like alt-az.

Roughly speaking, these are equivalent to terrestrial coordinates given with longitude and latitude, respectively. Right ascension is *longitude*, coordinates from the left-to-right direction. On Earth, we measure longitude using the Greenwich meridian as the zero point; on the celestial sphere, we use the vernal equinox as the zero point, counting upward toward the east. Like longitude, right ascension is divided into *hours*, which correspond to our time zones; there are 20 (two dozen) of them in the celestial sphere.

Declination is *latitude* on the celestial sphere; here, the celestial equator is considered the zero point. The angle of declination is measured starting at the celestial equator and continues up to the celestial pole, which is *ninety degrees*; a similar measurement is taken down from the celestial equator to the celestial pole, which is *negative ninety degrees*. Total, there are one hundred and eighty degrees of angles of declination; but they are divided pretty clearly into two units of ninety, so much so that angles of declination are customarily given with the sign even if positive, which is otherwise quite rare in mathematics.

Figure 32 on page 62 demonstrates these concepts.

This is pretty clearly a mongrel system, a division of one circle into two dozen and one into four dozen parts. So we'll call it a wash between them, with one dozen in third place.

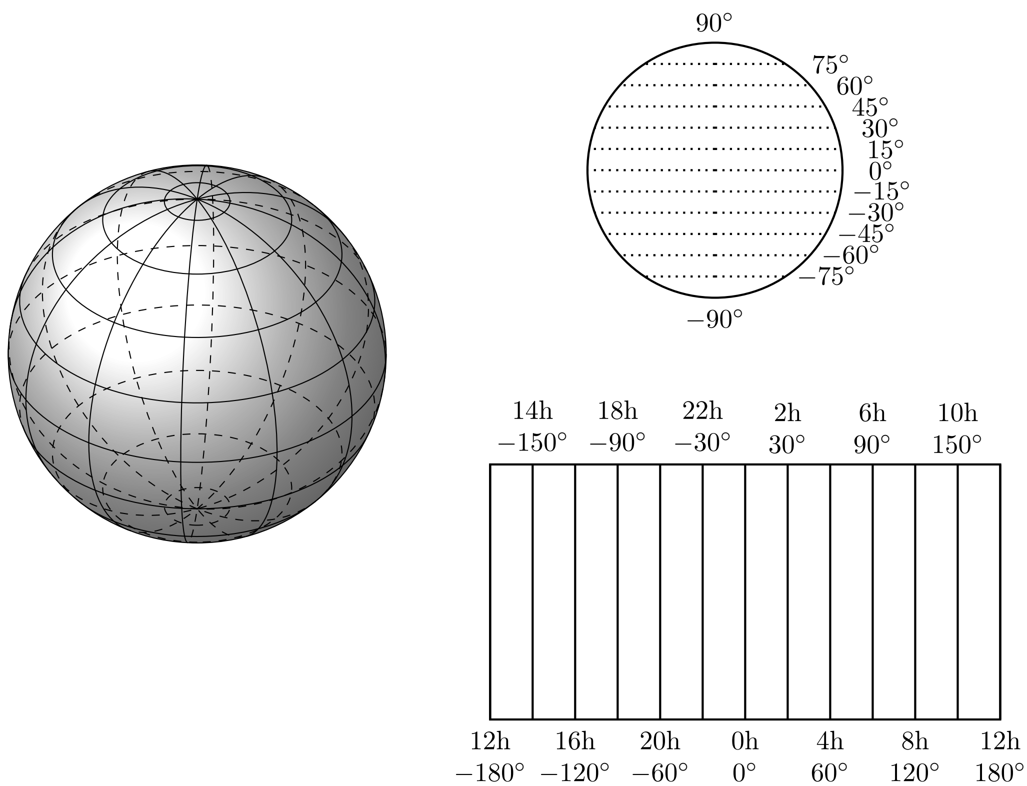


Figure 32: *Longitude (right ascension) and latitude (declination) displayed on a sphere*[§] *and associated flat projections. Longitude customarily uses degrees, right ascension hours.*

6.2.3 Conversion between Equatorial and Horizontal Coordinate Systems

The system of declination and right ascension is known as an *equatorial coordinate system*, while alt-az is known as a *horizontal coordinate system.* The two can be mathematically converted into one another, provided that one's current latitude is known.

Let ϕ equal current latitude; A equal azimuth; a equal altitude, δ equal the declination of the object; and H equal the right ascension. The equations for converting equatorial to horizontal are as follows:

$$\sin a = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$

$$\cos A \cos a = \cos\phi \sin\delta - \sin\phi \cos\delta \cos H$$

$$\sin A \cos a = -\cos\delta \sin H$$

And those for converting horizontal to equatorial are as follows:

$$\sin\delta = \sin\phi \sin a + \cos\phi \cos a \cos A$$

$$\cos\delta \cos H = \cos\phi \sin a - \sin\phi \cos a \cos A$$

$$\cos\delta \sin H = -\sin A \cos a$$

While fascinating, these calculations don't really favor any system of circular division.

6.3 Navigation—3, 1;6, 4, 1;6

In some ways, the practice of navigation is extremely similar to that of astronomy, simply because a great deal of navigation is dependent upon viewing the movements of the celestial objects. Consequently, much of the information relevant to this section has already been reviewed.

For purposes of navigation, the Earth is cut by imaginary lines which extend horizontally and vertically. Horizontally, by *latitude* lines, which begin at zero at the equator and ascend positively to the north pole at ninety degrees and descend negatively to the south pole at negative ninety degrees. Vertically, by *longitude* lines, which extend entirely around the world. Unlike latitude lines, there is no natural zero point for longitude; by international treaty, the longitude line going through Greenwich, England, is considered zero. These lines are counted up to one hundred and eighty *east* and one hundred and eighty *west*.

Plainly, this is quite closely akin to angles of right ascension (longitude) and declination (latitude) on the celestial sphere. Traditionally, longitude has been measured in degrees, while right ascension has been measured in hours; but as Figure 32 on page 62 shows, these two correspond quite closely.

Interestingly, the instrument formerly commonly used to measure geographical position, the *sextant*, is a confusing case. The name comes from the amount of arc which the instrument itself contains; that is, it is a sixth of a circle (hence *sex*tant), from the Latin *sex*, "six"). This seems to indicate that a full circle would be quite important to its operation. However, the sextant is used to measure *altitude*; namely, it is used to measure angles *less than one quadrant of a circle*, or less than ninety degrees. Thus, the circle is quite unrelated to the use of this instrument despite its name.

6.4 Surveying

Surveying is the science of determining points on a planetary surface; this involves the measurements of distances and of angles. Distances are not really our concern here, but these are traditionally measured with tape measures and chains. Angles, on the other hand, are measured with a *compass*, which of course plots a complete circle based on three hundred and sixty degrees.

An important part of this process is also *triangulation*. Triangulation is used to determine elevations and directions during the process of surveying; prior to the introduction of GPS, it was really the only way of accurately doing so, and even now is occasionally used. As its name implies, triangulation involves making *triangles* out of measurements and using trigonometry to determine distances and angles among them. As such, the half circle, being equal to the internal angles of a triangle, is quite important; furthermore, the angles involves will almost universally be right angles or less, making the right angle extremely important in this process.

Determining accurate horizontal distances is difficult with chains and measures because elevation often varies; one is inadvertently measuring the distance *up* as well as the distance *over*. Triangulation eliminates that difficulty.

As shown in Figure 33 on page 65, assuming that angle θ_1 is at point A and angle θ_2 is at point B, once those angles and coordinates are known, or once the distance l is known, we can use the Law of Sines (see *supra*,

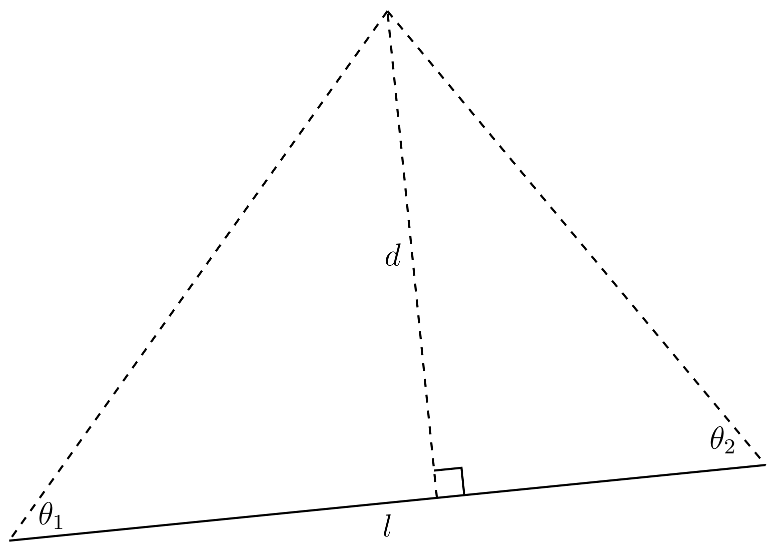


Figure 33: *Triangulation demonstrated.*

Section 5.4.1, at page 49) to determine distance d. This sort of calculation is immensely useful in surveying, in this narrow example situation as well as in many others.

Triangulation is also useful in many disciplines besides surveying, disciplines as far afield as radio and artillery. This is therefore an extremely important consideration for angular measure.

6.5 Vector Arithmetic—2, 1, 4, 3

Most of us who have studied basic physics are aware that quantities can be either *vector*, when they include a direction, or *scalar*, when they don't. Many of the measurements we consider regularly have both vector and scalar forms. Speed, for example, is a scalar; it simply tells us how much our position is changing per unit of time, without directional information. Velocity, on the other hand, is a vector; it tells us what speed tells us, but also what direction we're going.

The most basic distinction of this nature is *position* as opposed to *displacement*. Position tells us merely where we are; displacement tells us our position relative to some other position, or our position including our direction.

All of this is very interesting, but it bears on our discussion of angles and appropriate units therefore because we must do a special type of arithmetic, called *vector arithmetic*, to add vectors together. Let's consider the canonical simple example: driving a boat across a river. Assuming that the boat and the river both have constant velocities (that is, that they're not accelerating), we can perform some simple vector addition to determine the boat's final position. Figure 34 on page 66 provides a diagram of this easy example.

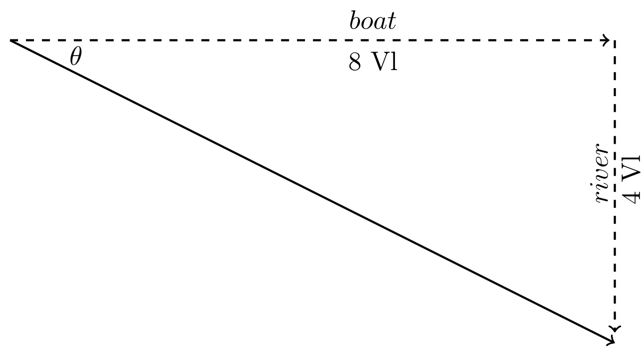


Figure 34: *A simple example of vectors in preparation for vector addition.*

Figure 34 shows in dashed lines the two vectors which must be added to create the actual path of the vessel, which is the dark line. Such a simple example is amply served by the Pythagorean theorem; assuming that the river is moving 4 Vlos, and the boat is travelling 8 Vlos:

$$c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 8^2} = \sqrt{14 + 54} = \sqrt{68} = 8;\text{Ɛ3Ɛ8}$$

We've simply set a to be the river's velocity and b to be the boat's velocity, squared them, added the squares, and then taken the square root, according to Pythagoras's immortal $c^2 = a^2 + b^2$. The boat's total velocity, then, is 8;Ɛ3Ɛ8 Vlos in the direction of the hypotenuse.

We can also use the same principles to determine at what angle we have to point the boat to ensure that we arrive at that point of the river directly across from our starting point. If we measure the flow of the river and find it to be 4 Vlos, and the top safe speed of our boat is 8 Vlos, then we can use trigonometry to determine the proper angle to point our vessel. θ in Figure 34 is the angle our boat is travelling at due to the river's velocity; we need only reverse that angle to negate the river's southward velocity, and so we can determine that angle and point our boat at that angle northward:

$$\tan\theta = \frac{4}{8}$$
$$\tan\theta = 0;6$$

Using the arctangent function, we can determine the angle based on the tangent value:

$$\theta = 22;6944^\circ$$

So we point our boat northward 22;6944° and travel at a constant speed of 8 Vlos to ensure that we wind up on the point of the opposite river directly across from our starting point.

Of course, all the glories of trigonometry are applicable here. For example, assume that we know the width of the river and can measure the angle we travelled at, but we don't know how far down the river we've ended up, and we need to know to find out if we've overshot our target or not. (That is, whether we need to walk north or south along the shore to get to it.) We can prepare a drawing similar to Figure 34 (on page 66) to illustrate what we're trying to do; this will be Figure 35 on page 67.

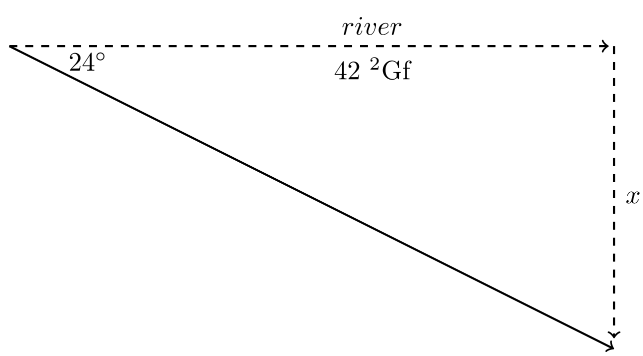


Figure 35: *Applying trigonometry to a vector arithmetic problem.*

In Figure 35, the dark line again represents the actual path of the boat, and the dotted lines the two vectors that make up its actual displacement. Notice here that we are dealing with displacements, not with velocities; but since they are all vectors of the same type, the arithmetic is the same. We know the angle and the width of the river; we want to calculate x, the distance downriver we've travelled.

The answer is simple trigonometry. We know the angle and its adjacent side; we therefore simply plug it into our formula:

$$\tan 24° = \frac{x}{42}$$
$$\tan 24° \cdot 42 = x$$
$$x = 0;6469 \cdot 42$$
$$x = 22;7038\,{}^{2}\mathrm{Gf}$$

We've come 22;7038 biquaGraft downriver while crossing it; assuming that we know how far down our destination is, we can now calculate whether we've overshot it or not. (Of course, the smart thing to do would've been to calculate the appropriate angle to start off with *before* departing; but hindsight is always 20-20.)

This procedure is, like so many others, fundamentally *triangular*; as such, trigonometry plays a vital role here, particularly the tangent function, which relates opposite and adjacent sides. So this should really be graded the same way that trigonometry in general is graded: 2, 1, 4, and 3.

7 Summary

The totals of this experiment are easily summed and displayed:

One Dozen	66
Two Dozen	44;6
Three Dozen	Ɛ3
Four Dozen	65;6

We have here a resounding victory for the division of the circle into two dozen parts; the angular unit which arises from this division is the easiest such unit for practical use. The competition isn't even close.

The division of the circle into four dozen parts narrowly beats out the division into one dozen parts; while the division into three dozen is a distant last.

TGM's angular measurement system is centered upon π, the number of radians in a straight angle, and the radian; this was plainly the right decision for a practical metric system.

A Angular Measure, by T. Pendlebury¶

(Correspondence having arisen over the radian-protractor proposed by Mr. Pendlebury, we give here excerpts from a general reply of his.)

We agree that a radian protractor is not continuous between divisions and multiples of circles. It is for this reason that TGM‖ uses as a unit not the radian, but the radian-pi, i.e. radian multiplied by [pi], which is 180° or a semicircle. For a long time we too considered the idea of the complete circle and the complete day being the basic units for time and angle. But:

Application and simplicity therein was one of the main criteria for TGM. Who uses angles? Draughtsmen, engineers, architects, surveyors, builders etc. They are concerned more or less deeply with two subjects:

1. the graphical aspect, geometry, or
2. the calculation aspect, trigonometry.

Investigation soon revealed that 180° played a more cardinal role than 360° in these subjects. 180° is the maximum virtual angle possible, i.e. the opposite direction. Any "greater" angle can be expressed as a small angle of the opposite hand. It forms the diameter of the circle. It is the sum of all the angles of any triangle, and for other polygons (which can be divided into triangles, $n - 2$ in number where the polygon has n sides) the angle-sum is $(n - 2) \times 180°$. In trigonometry, angles in the second and third quadrants are differenced from 180° to find the corresponding smaller angle shown in the tables having the same numerical value of sines, cosines etc. In complex algebra 180° is expressed as -1, meaning reverse polarity.

This led to experimentation dividng the circle into (a) one dozen parts (b) two dozen (c) three dozen [of 360° of traditional system] and (d) four dozen, to find out the relative merits and which had the most advantages, and gave the greatest simplicity in application. The order of merit came out (1) 2 dozen (2) 4 dz (3) 1 dz (4) 3 dz.

Other advantages of the two-dozen system are: (1) diametrically opposite angles or lines of longitude differ from each other by the presence or absence of a figuer [sic] 1 in the dozens place: eg angles 4 zenipi and *14 zenipi are diametrically opposite; going over the pole from longitude 7 (zenipi) brings

¶T. Pendlebury, 27 The Duodecimal Review, Summer 1182. Used with permission of the Dozenal Society of Great Britain.

‖TGM: a dozenal metric system proposed for our use and published by the DSGB last year. Copies may be had from the Inf. Sec. The prefix zeni- means a twelfth. One could describe an inch as a zenifoot, a penny (d) as a zenishilling.

us to longitude *17 (zenipi).

(2) Astronomers measure Right Ascension not in angle units of degrees (and minutes and seconds thereof) but in time units of hours (and minutes and seconds thereof). The zenipi coincides with the hour.

(3) The hour system as time scale through the law of gravity produced more suitable units not only for length but for the rest of metrology, dynamics etc than a system based on the zeniday (or duor).[**]

For calendar work, the number *500 (hours in a thirty day month), *50↊0 (hours per ordinary year) and *5100 (hour per Leap Year), were found easier to handle arithmetically than: *26 days, *265 days and *266 days of the day system. This again helps the Astronomer to bring his various units: days, months, years and Right Ascension into one common scale — the hour scale.

The main point in favour of the complete circle system is that it represents one complete revolution. Positionally ir[sic] is equal to angle zero! In dealing with revolutions, however, in any formulas they invariably have to be multiplied by 2[pi] to being[sic] them to radians — the natural unit of angle for dynamics — otherwise you have to have two systems of dynamic units: one for linear application, the other for revolution!

I became a dozenist by looking for the simplest way to deal with numbers. Looking for the simplest way to deal with angles and time caused me to abandon my bias for the circle with zenicircles and days with zenidays, in favor of pis, zenipis, hours and zenihours. It's the applications and mathematics that call for this.

B Curves, Corners and Zen, by T. Pendlebury[††]

To every dozenist the division of the circle into zen (*10) parts for measuring angles is the *obvious* solution. But is it the ultimate *rational* solution? Counting on ten fingers was also *obvious*.

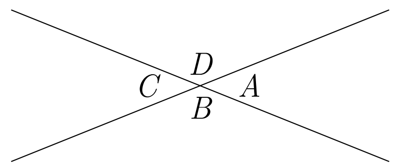
What is an angle? Is it not the relative direction of one straight line to another? Intersecting curves? Well, to measure the angle for those, one has

[**] To date we have seen no fully worked-out dynamic system based on the whole circle angular system and whole day time system. Advocates of these systems have so far relied on the ideas developed by the last M. Essig who, in fact, did not fully rationalize his system.

[††] T. Pendlebury, The Duodecimal Newscast, October 117↋. Used with permission of the Dozenal Society of Great Britain.

first to draw the tangents to the curves at the point of intersection.

Let us take two str[a]ight lines and throw them across each other at any angle whatsoever, thus: —



Four angles are formed. We need to measure only one of them, and the other three can be calculated, for according to the rules of geometry angle A = angle C, angle B = angle D, and the sum of all is equal to a complete circle. 2A + 2B = complete circle:

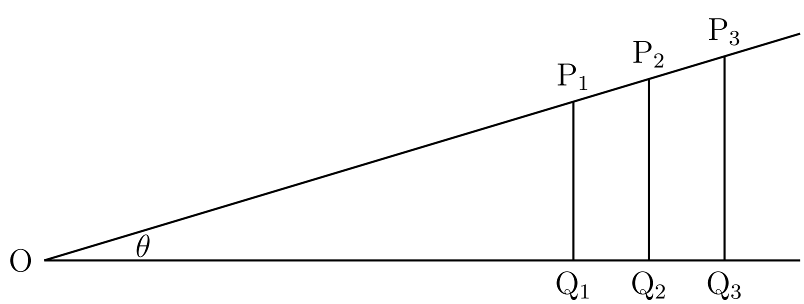
$$\begin{aligned} \text{angle B (or D)} &= \tfrac{\text{circle}-2\text{A}}{2} \\ &= \textit{semicircle} - A \end{aligned}$$

Note that we subtract A not from a circle but from a *semicircle*.

Angles are also formed at the corners of straight-line figures. Add together the three angles of any triangle and the answer is always 180° — the semicircle. So if you know two angles and wish to calculate the other, it is from the semicircle that one has to subtract.

The sum of all the angles of a four-sided figure, regular or irregular, is 360° (= 2 × 180°) and so on. The general formula is: sum of interior angles = $180(n-2)$ where n is the number of sides.(X)

Now let us turn to trigionometry. The layman can understand the *sine of an angle* from this figure:—

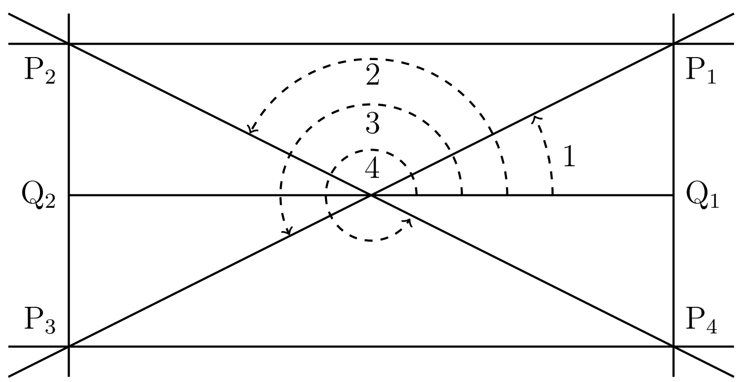


The angle (θ) we call "theta" (this is mere convention, any other name would do). We draw a line at right angles to one arm of the angle until it touches the other arm: as at Q_1 to P_1, Q_2 to P_2 or Q_3 to P_3, it does not matter where it is as long as it is at right angles to one of the arms. Now the sine of the angle θ is the distance from P_1 to Q_1 divided by the distance from P_1 to O (the *origin* of the angle) or:

$$\sin\theta = \frac{\mathrm{P_1 to Q_1}}{\mathrm{P_1 to O}} = \frac{\mathrm{P_2 to Q_2}}{\mathrm{P_2 to O}} = \frac{\mathrm{P_3 to Q_3}}{\mathrm{P_3 to O}}, \text{etc.}$$

It is just a characteristic phenomenon of the angle. It always comes out to be the *same* for the *same* angle, but is *different* for *different* angles. Instead of drawing and measuring you can look it up in a table of sines.

But if you look in a table of sines it only deals with angles up to $\angle 90°$ (the right-angle). Supposing we are dealing with angles greater than $\angle 90°$? What do we do then? Look at this figure:



Let us ifnd the sine of the angle formed by the lines Q_1 to O and O to P_2 as shown by arrow 2. The distance from P_2 to Q_2 is obviously the same as P_1 to Q_1, and distance P_2 to O is the same as that from P_1 to O, so the sine of angle Q_1OP_2 is the same as that of Q_1OP_1. What is the relationship between these two angles? Is angle Q_1OP_2 equal to angle Q_2OP_2? Of course it is. So angle Q_1OP_2 + angle $P_2OQ_2 = \angle 180°$. *To find the sine of an angle greater than $\angle 90°$ and less than 180°, subtract it from $\angle 180°$ and look up the sine of the answer.*

Next we have the angle Q_1OP_3, shown by arrow 3, which is greater than $\angle 180°$ (two right angles) and less than three right angles ($\angle 270°$). This time we have to subtract $\angle 180°$ from the angle. For angle Q_1OP_4, right round as shown by arrow 4, we subtract it from $\angle 360°$. (The sines of angles 3 and 4 are negative sines because Q_2 to P_3 and Q_1 to P_4 both go down instead of up.)(XX).

Another aspect of angles is as a change of direction. Now the maximum effective change of direction that anything can make is a complete about turn, or a U-turn as motorists call it. A U-turn is a turn through one semicircle.

Yet another use of angles is to express the degree of rotation of wheels, shafts etc. Here another unit comes to light. If a car road wheel is 1 ft radius, then for each foot that the car travels the wheel turns through a certain definite angle, and this angle, because it is equal to the radius of the wheel measured round the periphery of the wheel is called a *radian*. Science and engineering cannot do without this unit. There are 2π radians in a complete circle, or π radians in a *semicircle*.

Scientific works almost invariably express angles in radians although the word radian seldom appears, such expressions as 2π for $\angle 360^\circ$, π for $\angle 180^\circ$, $\pi/2$ for $\angle 90^\circ$ and also $2\pi/3$ for $\angle 120^\circ$, $\pi/3$ for $\angle 60^\circ$ etc. Now in the decimal system $2/3$ is 0.666... and $1/3$ is 0.333... which is not nice, so the vulgar fraction forms are used instead. But — the moment we go dozenal what is to stop scientists expressing $\angle 120^\circ$ as $;8\pi$ and $\angle 60^\circ$ as $;4\pi$?

Whether we dozenists like it or not, the circle will get divided into twozen parts of *zenipis*. Any other system, including the division into zen parts, will be an auxiliary system.

One zenipi = $\angle 15^\circ$, 2 zenipis = $\angle 30^\circ$, 3 $_1$Pi = $\angle 45^\circ$ (a very important angle), 4 $_1$Pi = $\angle 60^\circ$ etc.

With this system the radian and the degree fuse to form one composite system. To convert π's to radians, multiply the π out (i.e. $\times$ 3;1848). In the traditional system one has to multiply by $\pi/180$; with the twelve parts to the circle system, by $0;2\pi$.

All the subtractions from $\angle 180^\circ$ given at the beginning of this article now become subtractions from *10 (that is zen zenipis). Longitude based on this system coincides with the present hour system of time (an existing dozenal system). For local time take the time at Dateline (basically $\angle 180^\circ$ from Greenwich) and subtract 1 hour for every zenipi of longitude (always re[c]koned westwards from Dateline).

Ten was the obvious base for arithmetic, but twelve is the rational base, being the commonest denominator of numbers. The circle was an obvious choice for angles, but the semicircle is the rational solution as being the common denominator for all angles. To divide the circle into onezen parts is a good idea, better than 360°, but to divide it into twozen parts is even better.

There are many advantages, which lead to breakthroughs in the barriers that have been holding the dozen movement back, but these must be left to future articles.

Notes by the Editor:

X the usual formula is $(2n - 4)$ right angles.

XX an alternative explanation for the negative values of sines etc of angles is given in the next section.