Basic Arithmetic

written for

Adult Students

for working in the

Dozenal Base

Donald P. Goodman III

Dozenal numeration is a system of thinking of numbers in twelves, rather than tens. Twelve is a much more versatile number, having four even divisors—2, 3, 4, and 6—as opposed to only two for ten. This means that such hatefulness as "0.333…" for $^1\!/_3$ and "0.1666…" for $^1\!/_6$ are things of the past, replaced by easy "0;4" and "0;2."

In dozenal, counting goes "one, two, three, four, five, six, seven, eight, nine, ten, elv, dozen; dozen one, dozen two, dozen three, dozen four, dozen five, dozen six, dozen seven, dozen eight, dozen nine, dozen ten, dozen elv, two dozen, two dozen one…" It's written as such: 1, 2, 3, 4, 5, 6, 7, 8, 9, ꭒ, Ɛ, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1ꭒ, 1Ɛ, 20, 21…

Dozenal counting is at once much more efficient and much easier than decimal counting, and takes only a little bit of time to get used to. Further information can be had from the dozenal societies, as well as in many other places on the Internet.

The Dozenal Society of America
http://www.dozenal.org

The Dozenal Society of Great Britain
http://www.dozenalsociety.org.uk

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Introduction

Arithmetic is a term which too often strikes fear into the hearts of many, even well-educated and intelligent people. But fundamentally, arithmetic is not a difficult subject; it's a beautifully simple one, made up entirely of rules which work perfectly every single time and completely consistent within itself. To help show how marvelously simple this science truly is, we have decided to produce this text for adult students in the hope that the fear surrounding arithmetic might be dispelled and a love for numbers and numeracy be cultivated.

Before we begin studying arithmetic, however, it would serve us well to consider what exactly it is we will be studying, why we should bother, and how we should go about it. So let's consider these notions in turn. First, what is arithmetic's place among the sciences; second, why we should study it; and third, what our method will be in our exploration of it.

The study of quantity as quantity is called *mathematics*. Most basically, when we count things we are determining quantity, and when we consider that quantity independently of what we're actually counting (that is, when we consider *four* by itself rather than *four apples* or *four children*), we are studying quantity in its own right, which means we're studying mathematics.

Mathematics, being the study of quantity as quantity, deals with shapes, which we call *geometry*; and numbers, which we call *arithmetic*. (Some higher branches deal with greater abstractions, such as the manipulation of symbolic representations of unknown numbers, which we call *algebra*; but for our purposes geometry and arithmetic are enough.) The study of numbers, and how we manipulate those numbers by applying certain operations to them, is called *arithmetic*; and that is the topic of our study in this text.

Arithmetic is *not* "number theory," though for a long time the two were considered synonymous. However, number theory now refers to a broader field of study of which arithmetic is merely a part. Some of what we study in this text will touch on number theory, but we will never do it any kind of justice; our concern here is purely arithmetic, and not the more abstract studies of numbers.

Next comes the question, why should we bother? Anyone who has ever instructed students in any topic, particularly mathematics, has heard the plaintive cry, "But when will we use this in real life?" In vain does the instructor respond that utility is not the sole measure of value; that even so, the student will, in fact, use this knowledge in his "real life"; and so forth. But here, we write for *adult* students, who can approach these questions without the tired apathy of the adolescent. How do we answer this particular question—that is, why should we bother studying this topic—for arithmetic?

We should study arithmetic because it is a fundamental part of understanding the world around us. The physical world is made up of physical quantities; that is, quantities that can be counted, measured, and manipulated. To truly understand how we do these things—that is, count, measure, and manipulate—we must understand the rules by which these things are done. So it is necessary that we have at least a basic understanding of arithmetic.

Furthermore, arithmetic deals with abstract number, so it is itself an abstract science. As such, arithmetic operates by rules which work *infallibly* every time they are correctly applied. The study of arithmetic gives us an opportunity to see that logic of the physical universe applied in a perfect, ideal realm; we can see how the theories work without the mundane complications that inevitably attend all physical realities. This exercise can produce a deeper understanding of such realities than mere observation of those realities alone.

Finally, in this text we endeavor to approach this subject of arithmetic in the same way that it is approached in normal life. We begin with the most basic use of numbers: *counting*. We proceed from counting numbers to learning how to *write* those numbers, and then how to *speak about* them. Then we begin to consider the fundamental operations that can be applied to those numbers, and then approach the more advanced topics of manipulating quantities. Finally, we go back to the beginning and consider the methods and tricks that enable us to perform these basic operations quickly and mentally.

How to Use This Text

The first piece of advice is the most obvious: *read the text*. Great effort has been expended in explaining the many algorithms of arithmetic in a clear and unambiguous way. The student will, of course, be the judge of whether that effort was successful; but he cannot possibly judge this without actually reading it.

The second important element are the exercises. *Please do the exercises*, in longhand, on paper. Check them in Appendix A; many of the exercises have explanations of how the answer is obtained. Sometimes the exercises are the chief means of showing how to solve a particular type of problem; the algorithms for the various operations are not the end of the story, and the exercises give extra methods and strategies for addressing practical problems.

Finally, when the text suggests that something should be memorized, *please memorize it*. Many things in arithmetic are made much simpler by such memorization; and the methods of Part III require it. Memorization is a dull and joyless process, but it is a temporary difficulty that will pay enormous dividends in the future.

Using the print version of this text should be unremarkable; it is equipped with the usual amenities of print. References are listed by section and page number; a full and abbreviated table of contents makes it easy to locate individual topics; the glossary provides page numbers of references to given terms, as well as the definitions; and the Table of Definitions in Appendix D lists the pages where important concepts are actually defined.

The electronic version of the text has been designed to be as easy to use as possible. All cross-references and footnotes are links to their respective locations. Each exercise has a link to its answer in Appendix A. As far as possible, each word in the text which has a definition in the glossary (in Appendix C) is a link to that definition.

So we hope that this text proves as useful to the student as it proved to the writer in producing it.

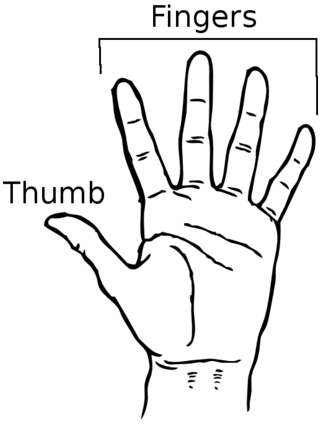
Part I
Counting and Numbers

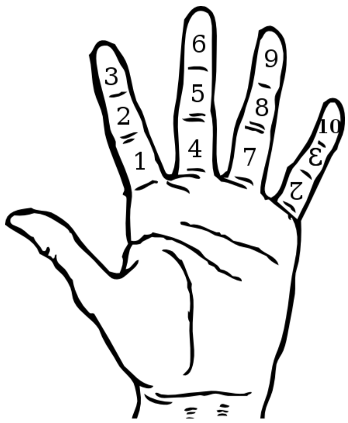
Chapter 1
Counting in Dozens

Learning to count in dozens is easy, but it does take some getting used to after having already learned to count in tens. So we will start at the very beginning, as if you'd never learned to count at all; soon you'll be counting dozens as quickly and easily—more so!—than you ever did in tens.

1.1 Counting from 1 to 10 on the Fingers

First, we count from *one* to *twelve*; and the easiest way to count is on our *fingers*. Hold up your hand, and you see that you have five *digits*. One of these, though, the thumb, is very different from the others; it branches from the side of our hand, not the top, and it's the only one that can touch all the others. So we say that we have four *fingers* and one *thumb*. Notice also that each finger has three segments; the thumb can touch all of those, too.





This lets us count to a *dozen* very easily on our fingers.

Those funny little scribbles on the palm picture to the right are called *digits* or *figures*; sometimes we simply call them *numbers*. We use them to count things; that is, we use them to record what number we've counted. Here's how.

Hold out your hand—right or left, it doesn't matter—with the palm facing you. We are going to count the segments on your fingers.

Start by placing the tip of your thumb on the first segment of your first finger, your *index finger*. Now say "*one*." You see in the picture how we write "one." There! You've counted off one of your finger segments! Now we can move on.

Now place the tip of your thumb on the second segment of your index finger. Say "*two*." Now you've counted *two* segments! But let's not stop there. Place your thumb on the top segment of your index finger; say "*three*." There again! You have *three* segments on your index finger! You know it, because you counted it!

But let's keep going. Do the same thing with your next finger, your *middle finger*. Count off "*four*," "*five*," and "*six*." Now you know how many segments are on your first two fingers! Move to the next, your *ring finger*. Count off "*seven*," "*eight*," "*nine*." Now finally your *pinky finger*, or *little finger*. Count off "*ten*," "*eleven*," and "*twelve*."

Congratulations! You've counted to a dozen! You can see how we write these numbers from the palm on the right, above. This is a big step; the first and hardest leap is over.

1.2 Counting from 10 to 100 on the Fingers

Now we can count even further, still using just our fingers. In Section 1.1, we counted from 1 to 10; now we will count from 10 to 100, numbers that for now we can call a *dozen* and a *gross*; and again, we don't even need to leave our fingers!

Start just as we did in Section 1.1, using either the right or the left hand, whichever feels more comfortable. Count from 1 (*one*) to 10 (*dozen*), just as before. While you do it, read the numbers off and look at the symbols we use for them below:

1	2	3	4	5	6	7	8	9	↊	↋	10
one	two	three	four	five	six	seven	eight	nine	ten	eleven	dozen

When you finish, you should have your hand in front of you, with your thumb pointing to the top segment of your pinky finger. Again, you've counted from 1 (*one*) to 10 (*dozen*)!

Now, take your other hand, whichever one you're not using, and put your thumb on the first segment of your index finger, just as you did when you were counting 1 (*one*). Remove your thumb from your pinky finger. Now you're pointing to the number 10 (one *dozen*)!

Your first hand counts *ones*; your second hand counts *dozens*. So the thumb on your second hand's index finger means 10; now you can keep counting up past 10 with your first hand, exactly the way you did when counting 1 to 10. Try it; count the segments on your first hand, while keeping your finger on the first segment of your index finger on your second hand. But each time you count, before you say the number, say *dozen* first; like this:

> Dozen *one*, dozen *two*, dozen *three*...

And keep going until you get to *dozen eleven*. You should now have your second hand still with your thumb on the first segment of the index finger, and your first hand with your thumb on the middle segment of your pinky.

Now, move the thumb on your second hand to the second segment of your index finger, and after *dozen eleven*, say *two dozen*.

You can then move your thumb on your first hand back to your index finger, and begin counting again, this time saying:

> Two dozen *one*, two dozen *two*, two dozen *three*...

What you're doing here is counting by *ones* with your first hand, and counting by *dozens* with your second hand. So on your first hand you keep counting as before:

1	2	3	4	5	6	7	8	9	↊	↋	10
one	two	three	four	five	six	seven	eight	nine	ten	eleven	dozen

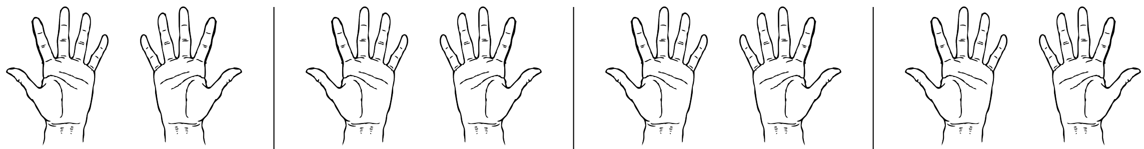
And every time you get to 10 (*dozen*) on your first hand, you count another *dozen* on your second hand, and move back to start again on your first. You can count by dozens like this:

10	20	30	40	50	60	70	80	90	↊0	↋0	100
one dozen	*two dozen*	*three dozen*	*four dozen*	*five dozen*	*six dozen*	*seven dozen*	*eight dozen*	*nine dozen*	*ten dozen*	*eleven dozen*	*dozen dozen*

You can count all the way through the dozens to *eleven dozen*, and then beyond! Eleven dozen one, eleven dozen two, eleven dozen three,. . . eleven dozen ten, eleven dozen eleven, and finally a *dozen dozen*, which sometimes we call a *gross*. All without ever leaving your fingers!

1.3 Counting Past 100: How Numbers Work

What we did in Section 1.2 by counting on our hands is the same thing we do when we're writing out figures; the first digit is the first hand, and the second digit is the second hand. We use the *place* of the number to show us what it means. Look at the chart below to see what we mean; in each part, the left hand is counting *dozens*, the right hand *ones*.



4	3	8	↊	↋	↊	9	6
Four dozens,	*three ones*	*Eight dozens*,	*ten ones*	*Eleven dozens*,	*ten ones*	*Nine dozens*,	*six ones*

We only have two hands, so we can't go further than that on our fingers. But with *figures* rather than *fingers*, we can go as far as we want; simply add another figure to the number. For example, if we want to count *dozens of dozens*, or *grosses*, we just add another figure to the number:

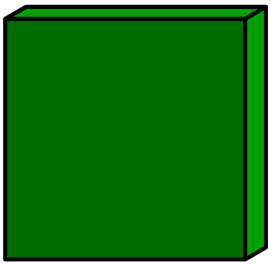
1	4	7	5	↊	3	8	2	8
One gross,	*four dozens*,	*seven ones*	*Five gross*,	*ten dozens*,	*three ones*	*Eight gross*,	*two dozens*,	*eight ones*

And we can go even further; if we want to count *dozens of grosses*, or *great-grosses*, we can simply add another figure; for example, 5↊74 is *five great-gross, ten gross, seven dozens, and four ones*.

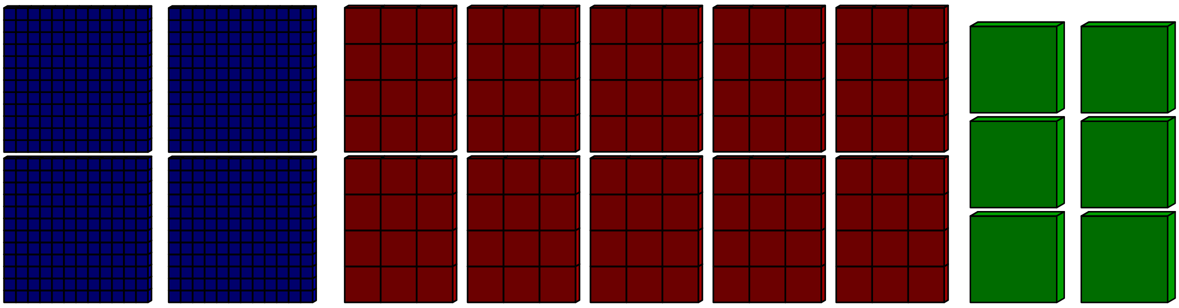
Usually we leave off the word *ones*, and so we can instead say "four dozen three."

Each new *figure* means the same as the previous figure, but multiplied by *twelve*, or the *dozen*. We can keep adding figures as long as we need to, to make as large a number as we need to.

Let's look at it another way: counting boxes. Blue has 100 boxes; red has 10 boxes; green is simply 1 box. Visually:



So let's look at a few groups of boxes and see how these translate to numbers.



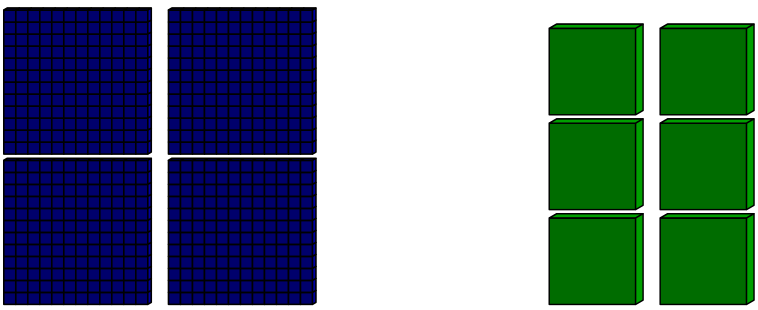
We can translate this into digits very directly: because there are 4 groups of 100, ↊ groups of 10, and 6 groups of one, we simply write them in:

4 ↊ 6

In other words, each digit in the number represents a number of *groups* of different sizes. The first digit represents groups of *ones*; the second represents groups of *twelves*; the third, groups of *grosses*; and so on, each digit representing groups of the same number as the last, multiplied by twelve.

This is represented visually here with three different sizes of group; one simply counts the groups of each, then puts that number in the appropriate column. So here we counted six groups of *one* (in green); ten groups of *twelve* (in red); and four groups of *gross* (in blue), then lined them up to read 4↊6.

But what if we have none of a certain type of box? Consider the following:



Here we have 4 boxes of a gross, and 6 boxes of one; but how do we write this? 46? No, because that looks like four groups of *twelve*, not four groups of a *gross*. What to do?

We use a *zero*:

> ***zero***
>
> a symbol meaning nothing, no items; neither positive nor negative; 0

We've already seen zeroes here and there, and haven't made much remark of them, because we're used to them and don't think about them much. However, the humble little symbol 0 is what makes the whole edifice of modern arithmetic work.

So we write the number we struggled with above as:

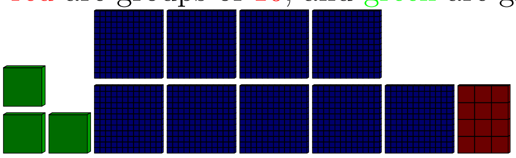
$$4 \quad 0 \quad 6$$

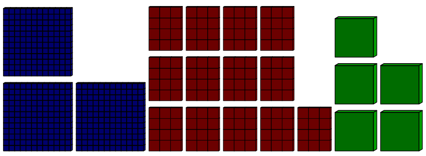
Four groups of a gross; no groups of a dozen; and six ones. The zero is the difference between a clear and consistent system and a jumble of numbers; it's what makes us able to identify the *place* of a number, and therefore its value, no matter what the number is.

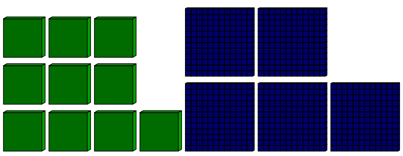
Some practice here will bear great fruit in later work.

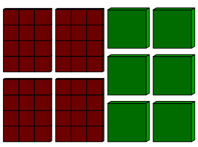
Exercises 1.1

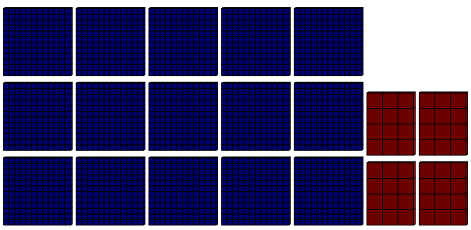
1. Write the number corresponding to the blocks below. Remember that blue are groups of 100, red are groups of 10, and green are groups of just 1. Answer

1(a)  Answer

1(b)  Answer

1(c)  Answer

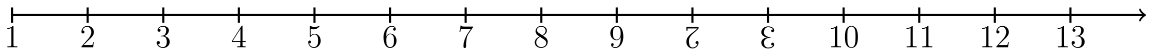
1(d)  Answer

1(e)  Answer

1.4 Zero and Beyond: Whole Numbers and Integers

Counting can go both ways. Usually, when we're counting, we want to count *forwards*; but sometimes we want to count *backwards*; and sometimes, we want to count things that aren't there. This seems confusing; but it's easy once you've seen it.

So far, we've been counting very simply, starting with 1 (*one*) and continuing as far as we need to go. We can put those numbers on a *number line*, like this:

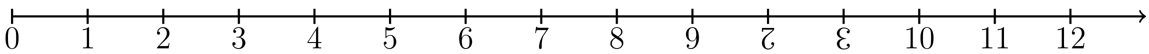


You can think of the vertical line on the left of the number line as meaning "the line starts here," and the arrow on the right as meaning "the line keeps going for as long as you care to take it." This is the type of counting we've been doing so far; and the sort of numbers we've been using to do it are called *natural numbers*.

natural number

the set of numbers used for counting and ordering real objects; counting numbers; equivalent to the set of all positive integers; $\{1, 2, 3 \ldots\}; \mathbb{N}^*$

But let's imagine we want to go a little further than this. We're counting things; but we want to be able to say that we have *none* of a certain type of thing before we start counting. We can do this with the symbol "0," which we call *zero*; it means, literally, nothing. When we add that in, our number line looks like this:

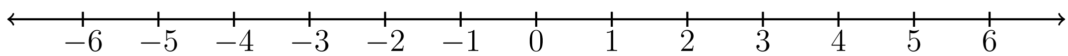


We call these *whole numbers*.

whole number

the set of natural numbers with the addition of zero; $\{0, 1, 2, 3 \ldots\}; \mathbb{N}^0$

But things can get even more interesting from here. We've already decided that we can count *zero* things; that is, no things, before we even get to one. What if we started counting *backwards*, to even less than none? It's impossible to have *less than zero* anything, of course, but this *concept* seems to make sense. Even though we can't physically *count* items with these numbers, we can *think* about them as though we can. This type of abstraction is very common in mathematics, even in basic arithmetic. So these numbers, by which we count backwards from zero, are called *negative numbers*, and we draw our number line, when we include them, like this:



As we can see, the *negative* numbers are marked different from the *positive* numbers (numbers greater than zero) by the sign $-$, called a *negative sign* or a *minus sign*.

We can also note that *now* our number line has an arrow on both ends. This is because not only can we continue counting *up* for as long as we want to, we can also continue counting *down* for as long as we want to; we will never run out of numbers either way.

When we include these negative numbers, as well as zero (which is neither positive nor negative), in our number lines, we say that we are using *integers*.

integer

Any number, positive, negative, or zero, that can be written without a fractional component. $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}; \mathbb{Z}$

Using the full range of integers allows us to do many interesting things. For example, we choose a specific date to start numbering our years; all years *after* that date we count in *natural numbers* (nothing but 1 through . . . well, through whatever you want), while all years *before* that date use *negative numbers*. Without the number line of integers, such devices as this would be impossible.

There remains one more interesting facet of counting things: how do we count *parts* of things? What do we do if we have *half* an apple?

1.5 Parts of Things

Sometimes, the things we count aren't whole; and sometimes we need to separate a whole thing into parts. How can we do that if we don't have a way of writing about parts of things, instead of only about whole things, like our integers?

fraction

a number expressing a quantity of equal parts, either less than, equal to, or greater than a single whole

These parts of things are called *fractions*, and they are quite commonly needed. Fortunately, over the long history of humanity we've developed ways of doing this. The two most common are *vulgar fractions* and *digital fractions*. We'll take these each in their turn.

1.5.1 Vulgar Fractions

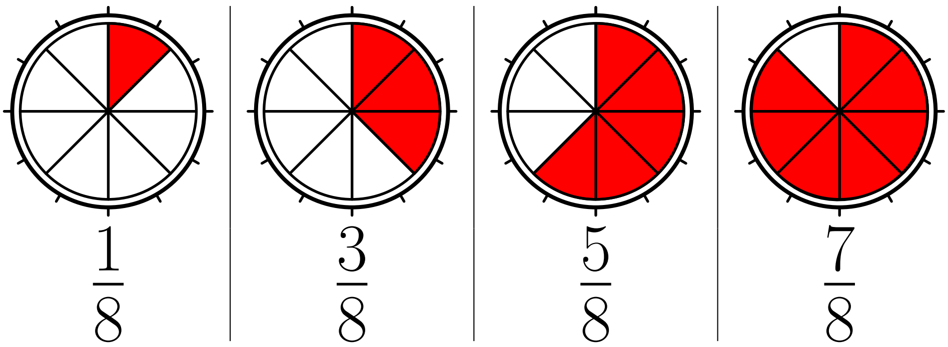
Vulgar fractions consist of writing out the number of parts in a single whole along with the number of parts we're actually dealing with, separated by a slash. Most of us are pretty familiar with these fractions; but we'll review the basics of them here.

vulgar fraction

a fraction expressed in terms of a number of parts above the number of those parts which make up a whole, separated by a straight line

First, we need to select how many parts we will have to the whole. We'll choose, entirely at random, *eight* for that number. It could be any integer at all, though.

Our vulgar fraction is then whatever number of parts we have grouped together with the number of parts in the whole (eight). We'll make some pictures here to demonstrate what we're looking at.



We have here four circles; it may be helpful to think of them as pies. Each circle is a whole pie. We divide each pie into eight pieces; so we say that *eight* is the number of parts in a whole. If we don't have the full eight parts, then we have a *fraction*. In the first circle, we have one of eight parts; in the second, three of eight; in the third, five of eight; and in the fourth, seven of eight. (Eight of eight, of course, would be simply 1.)

We've decided to write these out as *vulgar fractions*. We need two numbers for this: the number of parts in the whole, and the number of parts we actually have.

The former, the number of parts in the whole, we write on the bottom, and we call it the *denominator*.

denominator

In a vulgar fraction, the number of parts which make up a whole; placed at the bottom of a vulgar fraction

The latter, the number of parts we actually have, we write on the top, and we call it the *numerator*.

numerator

In a vulgar fraction, the number of parts of the whole which are actually present; placed at the top of a vulgar fraction

There are several ways we can read these fractions; let's take $\frac{5}{8}$ as an example. We can say that it's *five eighths*; we can say that it's *five out of eight*; we can say that it's *five over eight*. These are all equivalent ways of saying the same thing.

The numerator can even be *larger* than the denominator; for example, we might have $\frac{↊}{8}$, or *ten eighths*. This just means that we have *one whole*, or *eight eighths*; plus two more eighths, to make *ten eighths*. We could write the same number as $1\frac{2}{8}$, without changing the meaning.

There are names for these types of fractions. When the numerator and denominator are the same, of course, we have simply one; this is always true. When the numerator is *larger* than the denominator, we have an *improper fraction*; when the numerator is *smaller* than the denominator, we have a *proper fraction*.

When we express a vulgar fraction as a whole number combined with a proper fraction, we have a *mixed number*. For example, the fraction $\frac{5}{4}$ means that we have a whole divided into four parts, and we have in addition to that one fourth part of another whole. $\frac{5}{4}$ is an *improper fraction*; if we express it as $1\frac{1}{4}$, we have a *mixed number*. We will talk about how to convert improper fractions to mixed numbers and back again later; for now, it's just important to know which is which.

Vulgar fractions can express any fractional value *exactly*. Another word for these fractions is *ratio*, because they express the relationship of one number (the numerator, or parts actually present) to another (the denominator, or parts in each whole). We will learn a great deal more about ratios in Section 6.3.1, beginning on page 117; for now, it's just useful to know the term.

1.5.2 Digital Fractions

The other way to write fractions is to write them as *digital fractions*. Since we are counting in dozens, this simply means that we rewrite all fractions as fractions of *twelfths*.

> ***digital fraction***
>
> Fractions expressed inline, listing only the numerator, with the denominator inferred based on the place of the fractional part.

This is really much simpler than it sounds. Back in Section 1.3 on page 5, we saw how making numbers *bigger* was simply a matter of adding *digits* to them; whenever we finished counting out a *dozen* of ones, dozens, grosses, and so on, we simply added another digit and began counting again. Here, we do the same thing, but *backwards*; we add another digit to the *right* of the number, and this forms *fractional* parts.

You might be wondering how to tell the difference. Is a certain digit supposed to be one place *up*, or one place *down*? Well, we start with the digits to the *left*; these are the counting numbers, the *whole numbers* we know so well. Then, when we've finished saying the ones digit of the whole number, we put in a *radix point*, and then starting counting digits *down*.

The part to the left, the whole number, we call the *integer* or *integral* part; the part to the right, the fraction, we call the *fractional* part. For short, we often say *integrals* and *fractionals*.

For example, let's look at a pretty simple digital fraction:

$$4\;;\;6$$

Here, the semicolon, ;, is our *radix point*. The number to the *left* of the radix point is the *integral* part; in other words, the *whole number*. This means that we've counted out 4 units of whatever we're counting.

The number to the *right* of the radix point is the *fractional* part, which indicates some amount less than a unit. You'll notice that, although this is the fractional part of the number, it doesn't look like the vulgar fractions we saw in Subsection 1.5.1. That's because, with digital fractions, *we already know what the denominator is.* The denominator is always *twelve* (10), or twelve twelves, or twelve twelve twelves, and so on, as far as we care to take it.

So in this example above, we could rewrite this digital fraction as a vulgar fraction just by remembering that the denominator, for the first fractional digit, is *twelve.*

$$4;6 = 4\frac{6}{10}$$

Technically, we say either form as *four and six twelfths*; but more often, we pronounce the digital fraction as *four dit six*, pronouncing the radix point as "dit."

If we go on to another fractional digit, we can rewrite it as a vulgar fraction by remembering that the denominator is *twelve twelves*, or a *gross*:

$$4;6\text{Ɛ} = 4\frac{6\text{Ɛ}}{100}$$

Technically, again, this is pronounced as *four and six dozen eleven grossths*; but practically, we pronounce the digital fraction as *four dit six eleven*.

And we can take it to still another digit. This time, the denominator is going to be a *dozen dozen dozen*, or one *great-gross*.

$$4;6\text{Ɛ}9 = 4\frac{6\text{Ɛ}9}{1000}$$

Practically, this is pronounced *four dit six eleven nine*.

The pattern by now is obvious: the number to the *right* of the *radix point* is the *numerator* of the *fractional* part; the *denominator* of the *fractional* part is a *one* (1) followed by the same number of zeroes as there are digits in the fractional part.

Unlike vulgar fractions, digital fractions cannot express all fractional values exactly, because not all fractions fit into even parts of a dozen, a dozen dozen, or so forth. An example is $\frac{1}{5}$, which is 0;2497 2497 24..., repeating the digits "2497" infinitely, for as long as one wants to write them. Another example is *pi* (π), which is 3;1848 0949 3Ɛ..., following no pattern at all.

These two types of digital fractions are both called *nonterminating fractions*. The ones that follow a certain pattern, which repeats indefinitely, are called *repeating fractions*; the ones which do not are called *nonrepeating fractions*. *Irrational fractions* are specifically those *nonterminating fractions* which also do not repeat.

Why "irrational?" Because they aren't reasonable? Certainly not! We call them that because they are *not rational*; that is, not expressible as a *ratio* of two integers. We'll talk much more about ratios in Section 6.3.1,[1] far down the line from here; but for now, you can think of a ratio as an expression of the relationship between two numbers. So a *ratio*nal number is any which is a *ratio* between two integers:

rational number
any number which can be expressed as the ratio between two integers; all integers, along with all fractions with a terminating or repeating digital representation, are rational; $\mathbb{Q}$

Conversely, *irratio*nal numbers are those which are *not* ratios between two integers:

irrational number
any number which cannot be expressed as the ratio between two integers; only those numbers with digital representations that neither repeat nor terminate; $\mathbb{P}$

The set of all numbers including fractional parts, both rational and irrational, is called *real numbers*:

real number
The set of all numbers, from negative infinity to positive infinity, including all fractional parts; $\{\ldots-3,-2,-1,0,1,2,3\ldots\};\mathbb{R}$

Notice that each of these categories are *nested*; all natural numbers are whole numbers, all whole numbers are integers, all integers are rational numbers, and all rational numbers are real numbers.

And with that, we know as much set theory as we need for our study of arithmetic.

Exercises 1.2

2. Tell whether the following are natural numbers, whole numbers, or real numbers. Remember that some numbers may be more than one of these. Answer

2(a) 7 Answer **2(b)** 8 Answer **2(c)** $8\frac{1}{4}$ Answer **2(d)** 0 Answer
2(e) 0;4↊8 Answer

3. Tell whether the following are rational or irrational fractions, and if rational, whether repeating fractions or nonrepeating fractions. Answer

3(a) 7;6892 3 Answer **3(b)** 8;3333... Answer **3(c)** 2;8752 3606... Answer
3(d) ↋9↊4;2497 2497 2497 24... Answer **3(e)** ↊;689 Answer

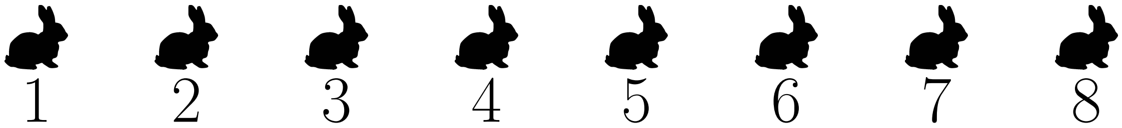
[1] *See infra*, Section 6.3.1, at 117.

1.6 Counting by Other Numbers

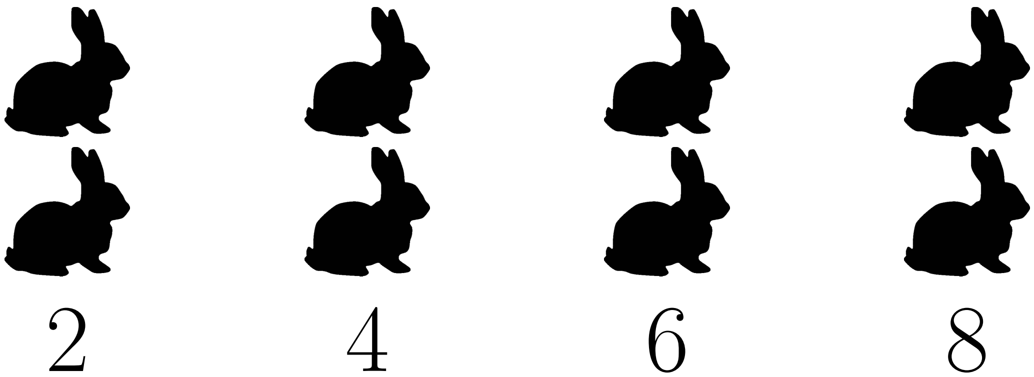
So far, we've only counted by *ones*; that is, we've always proceeded by one whole unit at a time. That's why we've counted 1, 2, 3,…

But in fact, we can count by any integer; and indeed, that's essentially what we're doing in *multiplication*, which we'll learn about later.[2] For now, we'll just consider counting by numbers other than 1 as a special type of counting, rather than as arithmetic.

So let's begin by counting by the next integer up from 1; namely, 2. First: why would we want to do such a thing? Let's consider counting *rabbits*. When we just want to know how many rabbits we have, we count them by ones:



And by doing this, we know that we have *eight* (8) rabbits. But we have our rabbits caged in *pairs*, and none of them have litters at the moment (an unlikely scenario for rabbits, but useful for illustration), so it would be easier to count by *twos* rather than by *ones*. In other words, rather than pointing at each cage and citing two numbers ("one, two"), we point at each cage and cite only one; but we increment by twos, like so:



We can count by twos for as long as we like, just remembering that we increase each count by *two*, not by one:

2, 4, 6, 8, ↊, 10, 12, 14, 16, 18, 1↊, 20, 22, 24, 26, 28, 2↊, 30…

We don't need to start at 2, either; we could begin counting by twos at 1:

1, 3, 5, 7, 9, ↋, 11, 13, 15, 17, 19, 1↋, 21, 23, 25, 27, 29, 2↋, 31…

[2] *See infra*, Section 5.3, at 51.

Or, indeed, at any number:

53, 55, 57, 59, 5Ɛ, 61, 63...

Nor must it be by twos. Say we increment each count by *three* rather than merely two:

3, 6, 9, 10, 13, 16, 19, 20, 23, 26, 29, 30...

And the same goes here; we don't need to start at 3:

87, 8Ϛ, 91, 94, 97, 9Ϛ, Ϛ1, Ϛ4...

This is simply normal counting, but we're inserting more than one step at each count. So when counting by two, we're skipping every other number. When counting by three, we're skipping two numbers for every one we count. When counting by four, we're skipping *three* numbers for every one we count. For example:

1, 2, 3, 4, 5, 6, 7, 8, 9, Ϛ, Ɛ, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1Ϛ, 1Ɛ, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2Ϛ, 2Ɛ, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 3Ϛ, 3Ɛ, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 4Ϛ, 4Ɛ, 50...

Above, starting at 1, in red we see the numbers we skip, and in blue we see the numbers we count. It's clear that, when counting by *fours*, we are skipping *three* numbers for every one we count. Another way of saying it is that we count every *fourth* number.

Similarly, when counting by *twos*, we skip one for every one we count; or, another way of saying it, we count every *second* number.

7, 8, 9, Ϛ, Ɛ, 10, 11, 12...

This type of counting is often very useful, and is also technically a type of *multiplication*, which means that we're already learning some more advanced arithmetic. Let's do a little practice in counting by numbers other than one, then, before we move on.

Exercises 1.3

4. Count by 5s from 43 to 77. Answer

5. Count by 3s from 32 to 95. Answer

6. Count by 2s from 0 to 9. Answer

7. Count by 8s from 34 to 54. Answer

8. Count by 10s from 50 to Ϛ0. Answer

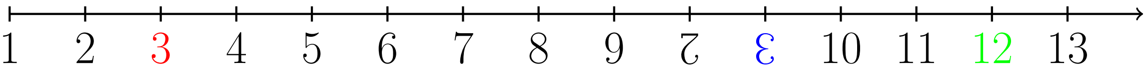
9. Count by 4s from 3 to 47. Answer

Ϛ. Count by 10s from 43 to Ɛ3. Answer

1.7 Comparing Numbers

It is frequently necessary not only that we count, but also that we *compare*; that is, that we determine the relative sizes of two or more numbers. This is mostly quite simple stuff; however, it is sometimes less than straightforward with vulgar fractions, and there is a specialized set of symbols that we use for this in mathematics, so it bears some study here.

The easiest type of comparing is that of *integers*. The easiest way to envision this is by setting up yet another number line. Consider the following, paying particular attention to the numbers highlighted in red, blue, and green.

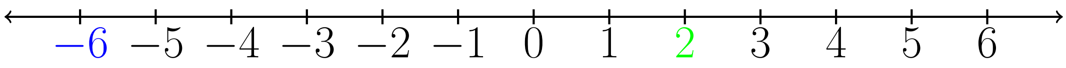


We need to make some judgments about the relationships between those highlighted numbers. We need to know whether they are *greater than*, *lesser than*, *equal to*, or *not equal to* one another.

$$3 < ↋ \qquad 3 < 12$$
$$↋ < 12 \qquad ↋ > 3$$
$$12 > ↋ \qquad 12 > 3$$

A note about the symbols: we use the symbols $>$ and $<$ to represent *greater than* and *less than* respectively; the sign always "points toward" the lesser number. So, to take an example from above, $↋ < 12$ can be read either "eleven is lesser than one-dozen-two" or "one-dozen-two is greater than eleven." Typically, however, these are read from left to right.

As we can see, numbers to the *left* of the number line are *lesser than* numbers to the *right* of the number line. That applies to negative numbers, too, even when this seems counterintuitive; consider the following:



Above, -6 is lesser than 2 ($-6 < 2$), even though $6 > 2$; this is because -6 is *left* of 2 on the number line.

Two integers are *equal* if their values are the same. We represent *equality* with the symbol $=$. *Inequality*, on the other hand, is a statement of *differing* values, and is expressed by the symbol $\neq$.

$$3 = 3 \qquad ↊ \neq -4$$

This leads us to two very important concepts:

equation

an expression containing a statement of equality, typically by the "=" sign

Equations are extremely important parts of mathematics. Sometimes they are very simple, like $1 + 1 = 2$; sometimes they are extremely complex, particularly in higher mathematics like algebra or calculus:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad f'(x) = \frac{f(x - \Delta x) - f(x)}{\Delta x}$$

These two equations, known as the *quadratic equation* and the *differential* respectively, are beyond the scope of a basic arithmetic text; they are merely demonstrations of what equations might look like.

inequality

the state of having different values; also, an expression containing a statement of inequality, including comparisons, typically using symbols such as $>$, $<$, or $\neq$

In mathematics, we often deal with *inequalities*, as well. Inequalities express relations between numbers other than equality; they include "greater than," "less than," and "not equal to," for our purposes.

$$||x + y|| \geq ||x|| + ||y||$$

The above inequality is one of the most important in geometry, the Triangle Inequality. Again, discussing it is really beyond our scope here; but it is an example of an inequality very important in mathematics.

Incidentally, it also shows us one of two other important comparisons for inequalities: "greater than or equal to," displayed as $\geq$. There is a corresponding "less than or equal to," which predictably looks like $\leq$. These mean exactly what they sound like they do.

Chapter 2
Words for Counting

Thus far we've used normal, traditional English words for numbers, like *dozen*, *gross*, and *great-gross*. These are fine words, and there's no reason for them to be abandoned. But the system does get to be difficult with very large numbers, when we must begin speaking of grosses of great-grosses and the like; and with very small fractions of numbers, where we must speak of grossths, great-grossths, grossths of great-grossths, and the like.

There *must* be a better way; and there is. We've already learned, in Chapter 1, how to count in dozens; now we'll learn how to talk about what we've counted, using Systematic Dozenal Nomenclature (SDN).

2.1 Basics of SDN

Systematic Dozenal Nomenclature is a simple system, much simpler than the hundreds, thousands, millions, and so forth that many people use. For its most common uses, it requires only that we learn twelve new words and two new suffixes.

We'll start with a table of SDN's new words. Do not be intimidated by this table; remember, it's just twelve new words, and you probably know most of them already.

Nil	*Un*	*Bi*	*Tri*	*Quad*	*Pent*	*Hex*	*Sept*	*Oct*	*Enn*	*Dec*	*Lev*
0	1	2	3	4	5	6	7	8	9	↊	↋

This table should be memorized before proceeding.

Now, we will learn two suffixes; one of these is for numbers which are *larger than one*, the other for numbers which are *smaller than one*; in other words, fractions.

Larger than 1	Smaller than 1
-qua	*-cia*

This short table must also be memorized.

Now, we've already seen that we can make numbers *larger* or *smaller* by adding *zeroes*; if we add the zeroes to the *left* of the *radix point* (the *dit*), it makes the number larger; while if we add it to the *right* of the *dit*, it makes the *fractional* part of the number smaller. To demonstrate:

Larger	Smaller
4↊8;532 ← 0	4↊8;532 ← 0
↓	↓
4↊80;532	4↊8;0532

Clearly, the zero on the left-hand side of the table makes the number *larger*, while that on the right-hand side of the table makes the fractional part of the number *smaller*. We can continue to make them larger or smaller, depending on where we put the zeroes, for as long as we like.

In SDN, we use this to our advantage. Let's take the very simplest case, composed of only a single digit. Let's say that the digit is 7. The table below shows us how we can use SDN in a simple way to refer to any size of number.

Larger		Smaller	
7	Seven	7	Seven
70	Seven un*qua*	0;7	Seven un*cia*
700	Seven bi*qua*	0;07	Seven bi*cia*
7000	Seven tri*qua*	0;007	Seven tri*cia*
7 0000	Seven quad*qua*	0;0007	Seven quad*cia*
70 0000	Seven pent*qua*	0;0000 7	Seven pent*cia*
700 0000	Seven hex*qua*	0;0000 07	Seven hex*cia*
7000 0000	Seven sept*qua*	0;0000 007	Seven sept*cia*
7 0000 0000	Seven oct*qua*	0;0000 0007	Seven oct*cia*
70 0000 0000	Seven enn*qua*	0;0000 0000 7	Seven enn*cia*
700 0000 0000	Seven dec*qua*	0;0000 0000 07	Seven dec*cia*
7000 0000 0000	Seven lev*qua*	0;0000 0000 007	Seven lev*cia*
7 0000 0000 0000	Seven unnil*qua*	0;0000 0000 0007	Seven unnil*cia*
70 0000 0000 0000	Seven unun*qua*	0;0000 0000 0000 7	Seven unun*cia*

(Note that we usually insert a small space, or sometimes a comma or some other mark, every four digits, to make these large numbers easier to read.)

Once again, do not be intimidated by this table; it's quite simple, really, just as simple as the counting we handled without any trouble in Chapter 1.

So we learned earlier the words for each number, 0–Ɛ. We simply count the number of zeroes we would have, and then we know the name of the number. If it's to the *left* of the radix point or dit, then we use the suffix *-qua* with the number; if it's to the *right*, we use the suffix *-cia* with the number. And that's all there is to it.

Obviously, most numbers we deal with won't be just one digit and a string of zeroes; there will be many digits mixed together. But this system still works. Take the leftmost number, determine its rank, and use that as your number word. The rest of the digits can simply be listed, since their role in the number is clear once the first rank is known.

How do you determine its rank? Simply count the digits other than the highest rank. For example:

3 5748 Ɛ872

Now, the highest rank here is, clearly, the leftmost 3:

3 5748 Ɛ872

Now, count the other digits:

3 5748 Ɛ872

The cyan digits above are, of course, eight; so we read the above number as:

Three *octqua* five seven four eight ten eight seven two

Let's try another number, this time using the *negative* orders of magnitude.

0;0000 05Ɛ4

Now, the highest rank is clearly the leftmost non-zero digit, 5:

0;0000 05Ɛ4

So we then count the other digits; in this case, the digits to the *left* of that highest rank, *including the zero before the radix point*:

0;0000 05Ɛ4

This number is clearly six; the whole number is therefore read as:

Five *hexcia* elv four

What about *mixed* numbers, though, which have both a whole number (*integral*) part and a fractional part? Well, these are quite simple; let's take an example:

3 Ɛ7Ɛ;85Ɛ6 4

Following the procedure above, let's read out the integral part:

3Ɛ7Ɛ;85Ɛ6 4

This give us three *triqua ten seven elv* for the integral part. Now, we simply read off the dit and list the digits, exactly as we do in decimal:

Three triqua ten seven elv *dit* eight five ten six four

This method can work even when there is no integral part, simply by stating the zeroes; e.g., pronouncing "0;005" as "zero dit zero zero five." But when there are many zeroes, it is likely easier to determine the rank.

Some further examples might be helpful.

Examples

1. Read off each whole number according to the SDN system.

1(a) 4Ɛ8
Four biqua ten eight.

1(b) 2852
Two trique eight five two.

1(c) 98Ɛ1 Ɛ9Ɛ4
Nine septqua eight eleven one ten nine ten four.

2. Read off each fractional number according to the SDN system. Do not use "dit," but find the rank and list the digits as above.

2(a) 0;056
Five bicia six.

2(b) 0;00Ɛ8 3
Ten tricia eight three.

2(c) 0;0000 047
Four hexcia seven.

3. Read off each mixed number according to the SDN system.

3(a) 56;891Ɛ
Five unqua six dit eight nine one ten.

3(b) 98Ɛ1 Ɛ9Ɛ4;Ɛ814 842Ɛ 9
Nine septqua eight eleven one ten nine ten four dit ten eight one four eight four two eleven nine.

3(c) 3961;4
Three triqua nine six one dit four.

When there are only two digits in the number, often *unqua* can be abbreviated simply to *-qua*; so Example 3(a) could instead be read as "Fivequa six dit eight nine one ten."

And now we can not only count as high as we want, or as low as we want; we can also easily refer to all numbers simply and consistently.

Exercises 2.1

1. Read off each number according to the SDN system. Answer

1(a) 9 Answer **1(b)** 14 Answer **1(c)** 78 Answer **1(d)** Ɛ2 Answer

1(e) 497 Answer **1(f)** Ɛ38 Answer **1(g)** 6 52Ɛ Answer **1(h)** 3 Answer

1(i) 8 Ɛ693 03Ɛ0 Answer **1(j)** 543 5Ɛ02 Answer **1(k)** 6Ɛ 4036 Answer

2.2 Writing Large Numbers

We have already seen that large numbers can be spoken about using SDN; we've also seen that numbers can be written by writing out their digits in a particular order. But sometimes this method seems difficult to read, and sometimes it seems even wasteful. Consider the number *seven unnilqua*:

$$7000000000000$$

This is a long string of zeroes, which serves no purpose other than to tell us that the 7 isn't just *seven*, but seven *unnilqua*. To determine this, one must count out the digits, and with a long string of zeroes this can often be difficult to do without losing count. Sometimes, too, we won't realize that we've lost count, and then we will read an entirely wrong number!

There are a number of ways to make reading such long numbers easier. The first we have already seen: put a small space, or commas, or some other mark every three or four digits. In dozenal, this is usually done every four digits, like so:

$$7\,0000\,0000\,0000$$

This small step makes it much easier to see what number we're dealing with; it's easy to count groups of four, and harder to lose track of the groups, which are larger in size and separated from one another, and therefore easier to distinguish.

Another way is often called *scientific notation*, more accurately called *exponential notation*; this consists of shrinking the number down to its most important part, then saying that the number equals that most important part times some other number. The above would be written thus:

$$7 \times 10^{10}$$

The part to the left of the "×" sign is called the *significand* or the *mantissa*, and the remainder is called the *exponent*. This method works quite well; but it's still rather bulky, and it requires arithmetical operations to properly interpret.

Another, shorter, and easier way is *marking orders of magnitude*, sometimes called *Pendlebury notation* after *Tom Pendlebury*, who first devised it. But first, let's define a term we've used a couple of times, to learn what it means:

order of magnitude

an approximate unit of size judged by powers of the number's base; so, e.g., a number approximately 10 times larger than another differs by one order of magnitude, 100 times larger differs by two orders of magnitude, and so forth

So 658 and 7297 differ by approximately one order of magnitude. Loosely speaking, orders of magnitude are simply the number of digits in the number.

Understanding this, Pendlebury notation consists of simply listing the order of magnitude (that is, how big the number is; or for short, how many zeroes you put) *before* the number. If the order of magnitude is *large*, we put it *superscripted*, like so:

$$7\,0000\,0000\,0000 = {}^{10}7$$

If, on the other hand, the order of magnitude is *small* (less than one), we put it *subscripted*:

$$0;0000\,0000\,0007 = {}_{10}7$$

There is no arithmetic necessary to interpret numbers written this way: they are simply the number of zeroes you add. This makes Pendlebury adscripting both very compact and very easy to read and understand.

Strictly speaking, though, this isn't just the number of zeroes to add; it's the number of places you move the *radix point*. For example, let's take a more complicated number:

$$7\,4875\,4000\,0000$$

Note that this number has the same number of digits as our number above; that is, it has the same *order of magnitude*. Now, remember that if we were to add a fractional part to this number, we would do it by adding a radix point to the end and affixing that fractional part. We don't really need one here; so let's just add zeroes, which will give us a radix point to work with but won't change the value of our number:

$$7\,4875\,4000\,0000;0000$$

The same number, with the same value; but a slightly different way of writing it. Now let's write that a little more simply by moving the radix point to the left, and marking how many places we moved it as we marked the number of zeroes above:

$$^{10}7;4875\,4$$

We say here that we're putting the order of magnitude "upstairs," to indicate that this is an *high* order of magnitude. We can write this same value more wordily in "scientific" notation as:

$$7;4875\,4 \times 10^{10}$$

The same concept applies backwards: when we're moving the radix point to the *right* instead of to the *left*, we put the number *downstairs*, like so:

$$0;0000\,0000\,0007\,4875\,4 = {}_{10}7;4875\,4$$

We can write this value more wordily in "scientific" notation, as well. Rather than putting numbers upstairs or downstairs, though, we put them upstairs, and make very small numbers *negative*. In other words, when moving the radix point to the *left*, we have a *positive* exponent; when moving it *right*, we have a *negative* exponent. This number, then, would be written as follows:

$$7;4875\,4 \times 10^{-10}$$

These two notations, *Pendlebury notation* (where the order of magnitude is expressed by putting the number "upstairs" or "downstairs") and *scientific notation*, provide compact and simple ways of writing very large, or very small, numbers without having to resort to long, difficult-to-read strings of digits.

However, there is an important difference between them. Pendlebury notation is a marker for *order of magnitude*; scientific notation is a marker of *multiplication and exponentiation*. It's not important what this means specifically at this stage; just remember that they are different. For example:

$$^{3}1 = 10^{3}$$

With Pendlebury notation, the number of digits the radix point is moved is equal to the number we put upstairs or downstairs; with scientific notation, it's the number put above and to the right of the 10 *minus one*. Be careful not to forget this distinction! Pendlebury notation is *not* exponential; it's a *substitute* for scientific notation.

Examples

4. Write the following numbers in *scientific notation* (also called *exponential notation*.)

4(a) 6000

Do the following:

1. If the number does not have a radix point, add one to it: 6000;.
2. Move the radix point to after the left-most non-zero number; count how many places and which way it's moved: 6;000; 3 places, left.
3. Write the factor and exponent (no need to worry what this means at the moment, just know what it's called): $6;000 \times 10^3$.
4. If you moved the radix point to the right, make the exponent negative: $6;000 \times 10^3$.
5. Erase any trailing or leading zeroes; if nothing is left after the radix point, erase that, too: 6×10^3.

The answer: 6×10^3.

4(b) 0;00872

Do the following:

1. If the number does not have a radix point, add one to it; we have one here: 0;00872.
2. Move the radix point to after the left-most non-zero number; count how many places and what directon it's moved: 0008;72, 3 places, right.
3. Write the factor and exponent: $0008;72 \times 10^3$.
4. If you moved the radix point to the right, make the exponent negative: $0008;72 \times 10^{-3}$.
5. Erase any trailing or leading zeroes, and the radix point if needed: $8;72 \times 10^{-3}$.

The answer: $8;72 \times 10^{-3}$.

5. Write the following numbers in *Pendlebury notation*, or *order of magnitude* notation.

5(a) 6000

Do the following:

1. If the number does not have a radix point, add one to it: 6000;.
2. Move the radix point to after the left-most non-zero digit: count how many places and in which direction it was moved: 6;000, three places, left.
3. Write the number of places *upstairs* if you moved the radix point left, and *downstairs* if you moved it right: $^{3}6;000$.
4. Remove any superfluous zeroes: $^{3}6$.

The answer: $^{3}6$.

5(b) 0;00872

Do the following:

1. If the number does not have a radix point, add one to it: 0;00872.
2. Move the radix point to after the left-most non-zero digit; count how many places and in which direction it was moved: 8;72, three places, right.

3. Write the number of places *upstairs* if you moved the radix point left, and *downstairs* if you moved it right: $_{3}8;72$.
4. Remove any superfluous zeroes: $_{3}8;72$.

The answer: $_{3}8;72$.

Exercises 2.2

2. Write the following numbers in *scientific notation* (also called *exponential notation*). Answer

2(a) 9 0000 0000 Answer **2(b)** 0;0000 000↋ Answer **2(c)** 0;↊842 5 Answer
2(d) 9 05↋↊;492↊ 386 Answer

3. Write the following numbers in *Pendlebury notation*. Answer

3(a) 9 0000 0000 Answer **3(b)** 0;0000 000↋ Answer **3(c)** 0;↊842 5 Answer
3(d) 9 05↋↊;492↊ 386 Answer

4. Expand the following to their full form. Answer

4(a) ^{4}↊ Answer **4(b)** $6;0 \times 10^8$ Answer **4(c)** $_{6}8;$↋34↊ Answer
4(d) $8;4 \times 10^{-4}$ Answer

2.3 Using SDN

Using SDN can extend well beyond merely reading off numbers. The system is quite versatile and can be applied to many different fields, including things as far afield as vehicular names, polygons, anniversaries, and many more.

2.3.1 Anniversaries

We speak about anniversaries quite a lot, and sometimes we need to refer to them not by number, but by name. E.g., many towns in America have celebrated one hundred and fiftieth or two hundred and fiftieth anniversaries; and the United States itself, only a few dozen years ago, celebrated its two hundredth anniversary.

But what are these anniversaries called?

The two hundredth anniversary was easy enough; that was the *bicentennial*. But these others are not so easy. Canonically, one hundred and fifty years is a *sesquicentennial*, and two hundred and fifty is a *sesquibicentennial*. There are, technically, words for nearly all anniversaries (or at least those divisible by five and ten), such as the *quadrancentennial* (twenty-five), the *dotranscentennial* (seventy-five), the *quasquicentennial* (one hundred and twenty-five), and the *dotransbicentennial* (one hundred and seventy-five).

In fact, several institutions, like Princeton University; the town of Reading, Pennsylvania; and Washington and Lee University have used the term *bicenquinquagenary* for their two-hundred-fifty year anniversary; but literally, it means a *ten thousand year* anniversary!

It would be exaggerating to say that there is *no* system behind these terms; but what system is there is so incredibly baroque that only a Latin or Greek scholar could really learn to use and appreciate it.

With all this craziness, is it any wonder that these words are rarely used except on banners once every quarter century, if that?

SDN offers an easy, direct, and consistent replacement for *all* this rigmarole. Simply form the number you wish to express as usual; then add the suffix *-ennial*, which will show that you're clearly referring to an anniversary.

So the ones that are easy now remain easy; e.g., "200th anniversary" is a *biquennial*, or a *binilnilennial* (however you prefer to say it). But the ones that are so difficult now, like a seventy-fifth or a hundred-fiftieth, are just as easy; e.g., "75th" is *septpentennial*, and a hundred-fiftieth is a *unpentnilennial*.

(These are dozenal equivalents, of course; but even direct decimal anniversaries are still easy. For example, a twenty-fifth (in dozenal 21) is a *biunennial*.)

Let's see more directly how this works.

Consider a town which has now existed for 160 years (that's one biqua six years, or in decimal 216 years). They are planning a large celebration (as befits this auspicious anniversary) and need to know what word to put on their banners for the occasion. So they group out the numbers like so:

1	6	0

They then apply the SDN words for each number:

1	6	0
un	*hex*	*nil*

Then, they add the suffix *-ennial*, just as we do to our familiar words "bicentennial" and the like:

1		6		0		years
un	-	*hex*	-	*nil*	-	*ennial*

And we are left with the proper word: it is the town's unhexnilennial.

This system works even with less round numbers. Take, for example, the United States of America. At the time of this writing, it is nearing the fourth of July, at which time it will be the 179th (decimally, 237th) anniversary of this country. In our current idiom, naming this anniversary is absurdly complex; but SDN makes it easy:

1		7		9		years
un	-	*sept*	-	*enn*	-	*ennial*

The country's unseptennennial.

Nor do larger numbers present any difficulty. For example, in Roman times, years were often numbered *ab urbe condita*, from the founding of the city (meaning, of course, the city of Rome). Traditionally, this was held to have occurred on 19 April, 529 BC (decimally, 21 April 753 BC). That was 1726 (decimally, 2766) years ago. So on 19 April 11Ɛ9, Rome had its

1		7		2		6		years
un	-	*sept*	-	*bi*	-	*hex*	-	*ennial*

unseptbihexennial. Happy birthday, Rome!

2.3.2 Periodicals

Similarly, there is a great deal of confusion regarding periodicals with our current numerical nomenclature. Is a newspaper published every two weeks a *biweekly*? Or is that a newspaper published twice a week? Or is that a *semiweekly*? And what about one published twice a month, regardless of weeks? Or once every two months? Or three times a year? Or seven times a year?

SDN, predictably, fixes the mess just as thoroughly as it does others.

Rather than -ennial, we can affix nearly any time-related suffix to SDN roots to indicate the period we're talking about. Take our every-two-weeks newspaper, for example:

2		
bi	-	*weekly*

Not coincidentally, SDN produces the term *biweekly*, which is the correct term in normal English (though it's unfortunately often misapplied to that which occurs twice a week, which is properly called *semiweekly*).

What about another newspaper, published twice a week? Well, in this case we can use *dit*, just as we do when we're reading numbers. Remembering that 0;6 (zero dit six) is one-half, and that twice a week means every half week, we get:

;		6		
dit	-	*hex*	-	*weekly*

And it's that simple.

It can even get more specific. Imagine a company which releases its newsletter seven times a month, perhaps selecting this unusual interval for symbolic reasons. In dozenal, as in decimal, a seventh is an irregular digital fraction, an unwieldy 0;186↊35 continuing to infinity; the construction we saw above, with dit, therefore isn't really suitable.

So SDN offers another particle, *per*, which is useful for such nonterminating fractions. So our newsletter is

1		/		7		
un	-	*per*	-	*sept*	-	*monthly*

unperseptmonthly. It's unlikely, of course, that this sort of term will be needed; but SDN makes it easy in case it is.

Examples

6. Give the SDN names for polygons, from the triangle (3-sided) to the icosagon (18-sided). The suffix indicating a polygon is, just as in normal English, *-gon*.

6(a) Triangle (3)
Trigon ("triangle," meaning "three-angled", is also perfectly orthodox SDN).

6(b) Rectangle (4)
Quadragon (of course, different words for specific types of quadragon, like "square" and "parallelogram," are still useful).

6(c) Pentagon (5)
Pentagon

6(d) Hexagon (6)
Hexagon

6(e) Heptagon (7)
Septagon

6(f) Octagon (8)
Octagon

6(g) Enneagon (9)
Ennagon

6(h) Decagon (Ɩ)
Decagon

6(i) Undecagon (Ɛ)
Levagon

6(j) Dodecagon (10)
Unquagon; *unniligon*

6(k) Triskaidecagon; tridecagon (11)
ununagon

6(l) Tetrakaidecagon; tetradecagon (12)
Unbigon

6(m) Pendedecagon (13)
Untrigon

6(n) Hexdecagon (14)
Unquadragon

6(o) Heptdecagon (15)
Unpentagon

6(p) Octdecagon (16)
Unhexagon

6(q) Enneadecagon (17)
Unseptagon

6(r) Icosagon (18)
Unoctagon

Exercises 2.3

5. Write the proper names of the five regular convex polyhedra in SDN. These polyhedra are, in current parlance, the *tetrahedron* (4 faces); the *cube* (6 faces); the *octahedron*

(8 faces); the *dodecahedron* (10 faces); and the *icosahedron* (18 faces). The suffix for polyhedra is *-hedron*. Answer

5(a) Tetrahedron Answer **5(b)** Cube Answer **5(c)** Octahedron Answer

5(d) Dodecahedron Answer **5(e)** Icosahedron Answer

6. Write the proper names of the following intervals. Answer

6(a) Twice a month. Answer **6(b)** Quarterly (four times a year). Answer

6(c) Every third week. Answer **6(d)** Three times a year. Answer

Chapter 3
Roman Numerals

The digits that we have been describing—0, 1, 2, 3, 4, 5, 6, 7, 8, 9, Ƨ, and Ɛ—which are combined in the ways that we've seen above,[1] are the most commonly used forms of digits today, by far. But they are certainly not the *only* form. In this chapter we will examine another way of writing numbers that is still quite common: *Roman numerals*. We'll see how these have been used historically and how they can be adapted to be used with dozens.

The means of writing numbers that we've learned, and that is used most of the time all over the world, is called *place notation*:

> ***place notation***
>
> a system of writing numbers in which each digit's value depends on its place among the other digits

While really any set of digits would work for this system, the digits we use are called *Hindu-Arabic*, due to the part of the world they first developed in, and the part of the world through which they came to Europe. This is the system which most lends itself to compact, unique identification of all possible values, and which gives the best algorithms for doing arithmetic.

Roman numerals, on the other hand (along with similar but still different systems, such as Greek and Hebrew numerals), operate on an entirely different basis. Since these systems are more limited (though not entirely limited) to decimal than place notation is, we will examine the system of Roman numerals on that basis first.

3.1 Decimal Roman Numerals

Roman numerals have been used entirely in decimal since their inception; it's necessary to continue thinking about them in decimal as we learn how they work. Later, we will see a way of using Roman numerals in a dozenal way.[2]

Rather than assigning each digit below a certain number (Ƨ, in decimal, or 10, in dozenal) a specific value, Roman numerals assign a symbol to certain important values and combine them in that way. The benefit of this is that there are fewer digits (though not much fewer); the detriment is that larger numbers become quite unwieldy, and fractional values are simply impossible, and must be represented by a different system.

For example, here are the figures in the Roman system:

Fig.	Doz.	Dec.	Fig.	Doz.	Dec.
I	1	1	C	84	100
V	5	5	D	358	500
X	Ƨ	10	M	6Ɛ4	1000
L	42	50			

[1] *See supra*, Section 1.3, at 5. [2] *See infra*, Section 3.2, at 29.

Notice the most salient feature of this system: *there is no zero.* Because the place of the individual figure in the number does not change that figure's value, it makes no difference what order the figures come in. (It is customary, however, to list larger values before shorter ones.) Since the order of the figures makes no difference, there is no need for a zero to ensure that figures remain in the right place.

In the system's simplest form, we then simply aggregate these figures to build up larger numbers. E.g., to make the value "13", which in decimal is written "15," we combine the figures for 5 and the figures for ↊ (decimal "10") to make a total of 13 (decimal "15"). In decimal:

$$\begin{array}{ccc} 10 & & 5 \\ \mathrm{X} & & \mathrm{V} \\ & \searrow \quad \swarrow & \\ & \mathrm{XV} & \end{array}$$

To fill in the gaps between these numbers, we simply repeat one of the lower numbers until it's the right value. For example, 8 falls between 5 and ↊, so we need to repeat the symbol for 1 when we write it:

$$\begin{array}{ccc} 5 & & 3 \\ \downarrow & & \downarrow \\ \mathrm{V} & & \mathrm{I\ \ I\ \ I} \\ \downarrow & & \downarrow \\ \mathrm{V} & & \mathrm{III} \\ & \searrow \quad \swarrow & \\ & \mathrm{VIII} & \end{array}$$

Any of the figures can be repeated this way, to form any (whole) number you wish. Note that you will never need to repeat a figure more than 4 times, however, since each level of figure is a multiple of 5. (If you don't know what this means yet, don't worry; we'll get to it in due time. For now, just trust us.)

This process can produce numbers of arbitrary size. For example, "1998" is "MDCCCCLXXXXVIII," a lengthy string of letters which is explained in the table to the right. As noted above, the system has few figures to remember, but results in very unwieldy numbers very quickly. Notice that, in our system (even in decimal!), "1998" only requires four figures, but 13 (dozenal) in Roman numerals!

M	1000
D	500
C	100
C	100
C	100
C	100
L	50
X	10
X	10
X	10
X	10
V	5
I	1
I	1
I	1
	1998

In an effort to shorten such numbers, a slightly more complicated form of the system developed. Rather than concatenating four of each figure together, this modified system concatenated only three; when four were needed, they inserted the next-higher figure, preceded by the current figure.

So in the original system, "4" was simply "IIII." This system is still seen on some old-style watch-faces and the like. In the modified system, rather than concatenating four Is, the next-higher figure ("V") was placed, preceded by the current figure ("I"). So "4" was not "IIII," but "IV."

Our "MDCCCCLXXXXVIII" then becomes "MDCDLXLVIII." A bit shorter, but not exactly manageable, and that much harder to read.

So by the original system, a clock goes I, II, III, IIII, V, VI, VII, VIII, VIIII, X, XI, XII. In the new system, it goes I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII.

It's easy to see why Hindu-Arabic numerals became dominant over Roman numerals; they are simply better for general numeral use, and we will see that they make the algorithms for arithmetic much easier. But since Roman numerals are still used in some formal and traditional contexts, it's still important to know how to use them.

Exercises 3.1

1. Write, in decimal Roman numerals (the newer system), the following decimal numbers. Answer

1(a) 4 Answer **1(b)** 14 Answer **1(c)** 19 Answer **1(d)** 26 Answer
1(e) 24 Answer **1(f)** 33 Answer **1(g)** 240 Answer **1(h)** 339 Answer
1(i) 1828 Answer **1(j)** 2016 Answer

2. Write, in Hindu-Arabic numerals, the following decimal Roman numerals. Note that these may be written in either the new or the old system. Answer

2(a) LIIII Answer **2(b)** LIV Answer **2(c)** MMXLVIII Answer
2(d) MXXIV Answer **2(e)** MCCCCLXXXXII Answer **2(f)** CXLIV Answer
2(g) MCDXCII Answer **2(h)** CXXXXIIII Answer

3.2 Dozenal Roman Numerals

While Roman numerals are archaic and seldom used, they are by no means obsolete. Such numerals carry a *gravitas* that our more mundane Hindu-Arabic numerals simply do not. Consequently, they continue to be used in many places, even though not in our daily arithmetic.

They are used in some places simply due to the heavy weight of long tradition; for example, movie publication dates continue to be listed in Roman numerals. (Ironically, the supposedly stodgier book industry has been using place notation for their publication years for ages.) They are used when we simply need numeration of a different appearance, such as in outlines or subsection numbering (where uppercase or lowercase can also be a distinction). They are used decoratively, on clockfaces and the like. And they are used whenever we need to add a certain dignity to something, such as the sequels to a dramatic series, or the numbers of kings of a certain name (who ever heard of Elizabeth 2?).

For biquennia, the Western world kept *all* of its accounts in such figures; we no longer do this, nor should we. However, there is still a place for Roman numerals in our modern world, and so the dozenal system needs to have some equivalent.

Adapting place notation for dozenal is easy: we need merely add ↊ and ↋, then step over every twelve steps rather than ten. With Roman numerals, however, we must actually change the value of the symbols themselves.

So rather than stepping up the figures at five and multiples of five, we step them up at six and multiples of six. Here's a table of Roman figures altered to have the appropriate values:

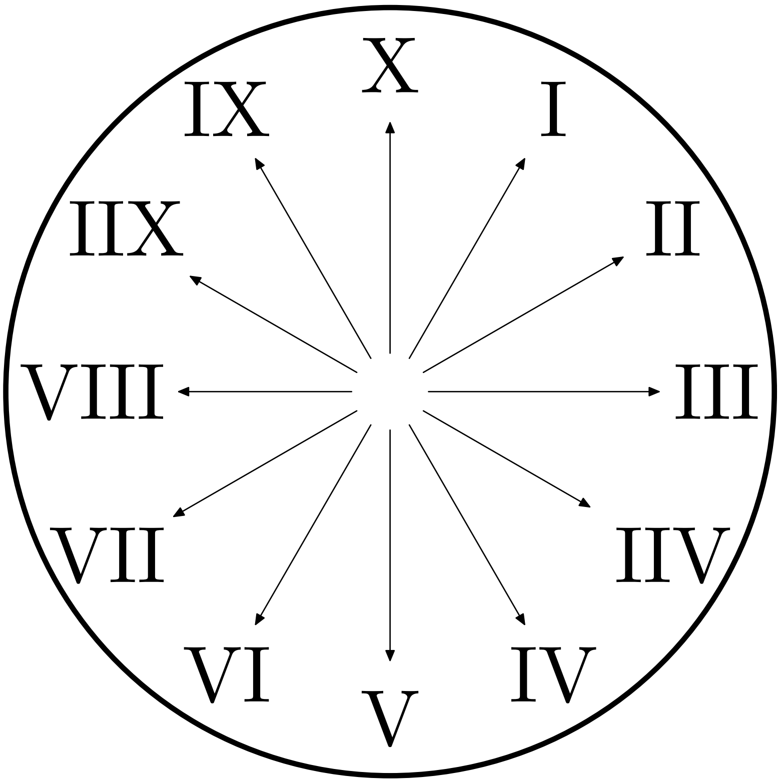
Fig.	Val.	Fig.	Val.
I	1	C	100
V	6	D	600
X	10	M	1000
L	60		

For combining these figures, we can precede a symbol by zero, one, or two of a smaller symbol; or succeed a symbol by zero, one, two, or three of a larger symbol; and in this way duplicate the breadth of the old, decimal system without requiring concatenation of more than three of the same figure. For example:

I	II	III	IIV	IV	V
VI	VII	VIII	IIX	IX	X
XI	XII	XIII	XIIV	XIV	XV
XVI	XVII	XVIII	XIIX	XIX	XX
XXI	XXII	XXIII	XXIIV	XXIV	XXV
XXVI	XXVII	XXVIII	XXIIX	XXIX	XXX

So the last numbered king of France (that is, the last before Louis-Philippe) was Louis XV (16); the one who got his head lopped off was Louis XIIV (14); but the current queen is still Elizabeth II, and the next Super Bowl (for the year 1201) is XXLIII (43).

This still leaves us with some quite nice-looking clock-faces, too, which can end properly at the round numeral X:



Notice the much nicer symmetry than we find in our traditional, decimal Roman-numeral clock. Numbers directly across the circle from one another all have the same number of figure-ones, and 6 and 10 stand directly across from one another in all their lone splendor. Furthermore, numbers which are the same distance from X on either side of the circle (e.g., "IIV" and "VII") are very clearly related, as well. After visualizing Roman numerals in this way, they come much more easily to the learner's mind.

Exercises 3.2

3. Write the following numbers (properly dozenal, this time) as dozenal Roman numerals. Answer

3(a) 4 Answer **3(b)** 14 Answer **3(c)** 19 Answer **3(d)** 26 Answer

3(e) 24 Answer **3(f)** 33 Answer **3(g)** 240 Answer **3(h)** 339 Answer

3(i) 1828 Answer **3(j)** 2016 Answer

4. Write, in Hindu-Arabic numerals, the following dozenal Roman numerals. Answer

4(a) LIII Answer **4(b)** LIV Answer **4(c)** MMXLVIII Answer

4(d) MXXIV Answer **4(e)** CCMXXLIIV Answer **4(f)** CXLIV Answer

4(g) MCDLXII Answer **4(h)** CXXLIIV Answer

Part II
Manipulating Numbers

Chapter 4
Introduction to Arithmetic

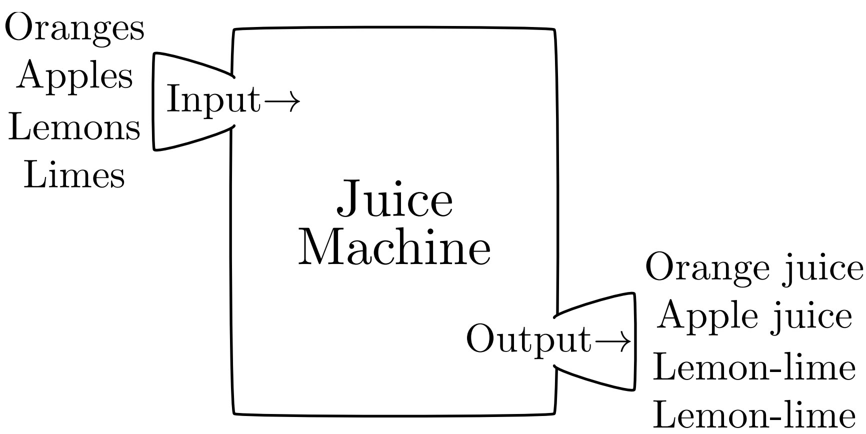
Now we move on to the actual use of all these numbers we've been counting with. It's been useful to review these basic concepts because most of us learned how to count in base ten, and to learn to count, and indeed think, in the superior base twelve is made easier by relearning all that we know. Also, these concepts are often only imperfectly learned, typically because only imperfectly taught, by us in the modern day, so it is useful to go back to the beginning and learn them more thoroughly.

Now, though, we will proceed to *arithmetic*; that is, the study of quantity and the manipulation of quantities. For our purposes, this consists primarily in the *four functions*: *addition*, *subtraction*, *multiplication*, and *division*. Other than these, though, we'll have to learn a little bit of new vocabulary to truly understand what we're dealing with here. It is truly just a little bit of vocabulary, and much of it will be divided up into the sections on the individual operations; however, it is necessary to be able to talk about what we're doing.

Primarily, we will be learning about *functions*; specifically, about *arithmetical functions*. We're all familiar with the concept of a function, but few of us are really familiar with the formal definition of it. Therefore, this will be our first vocabulary word:

> ***function***
>
> a relation between a set of inputs and a set of outputs such that each possible input is related to exactly one possible output

Consider a function like a juice machine; we throw in certain ingredients and can get out certain juices. But for each set of ingredients (inputs) we throw in, we can get out one, and only one, possible juice (output).



In the illustration above, we have four possible inputs: oranges, apples, lemons, and limes. We put these into our function, the juice machine. The juice machine then processes these inputs into outputs: orange juice, apple juice, and lemon-lime. The inputs are listed in the same order as their outputs; that is, oranges produce orange juice, apples produce apple juice, and both lemons and limes produce lemon-lime.

The thing to take away from this is that *every input has exactly one corresponding output*. Oranges won't produce both orange juice and apple juice; lemons won't produce both lemon-lime and orange juice. When you put a valid input (one of these four fruits) into the function, there is only one possible value you'll get out of the other end.

Notice that both lemons and limes produce lemon-lime as their output. This is fine; different inputs might produce the same output. But different outputs should never result from the same input; if this happens, then what we're dealing with is not a function.

For some more vocabulary, the group of possible inputs to a function is called the *domain*, and the group of possible outputs is called the *range*. However, that is part of a set of much more advanced mathematics that you really shouldn't worry about right now.

Our next word is an extremely important one: *algorithm*:

algorithm

a list of steps for calculating the result of a function from a given input

In other words, while the function tells us what we actually get, the algorithm tells us how we get it, in a finite series of steps. The word is most commonly used in computer science; however, algorithms are a necessary part of all of mathematics, particularly in the basic arithmetic that we are studying here.

Next we will learn a few words which relate to properties of various functions; this means that functions of a certain type will behave in certain ways. The following describe some particularly important ways that certain functions behave.

commutativity

a property of a function meaning that the order of the operands does not change the result; such a function we call *commutative*

This sounds much fancier than it really is; we'll see how simple it really is when we work on some commutative functions like addition and multiplication. The bottom line is simply that we can throw inputs at the function in any order without changing the resulting output; or, in other words, that order doesn't matter.

This should not be confused with our next concept:

associativity

a property of a function meaning that the order of operations does not change the result; such a function we call *associative*

While this sounds very similar to commutativity, it's really a different concept. *Commutative* functions don't care what order the *operands*—the inputs—are in; *associative* functions don't care what order the *operations*—the functions—are in. Some functions are *both*; e.g., addition. But they are still different things.

> ***anticommutativity***
>
> a property of a function meaning that the order of the operands changes the result such that the result of one is the negation of the other; such a function we call *anticommutative*

In other words, if we do the operation in one order, we get a number; if we do it in the opposite order, we get the negative of the same number. Obviously, given the name, this is precisely the opposite of commutativity.

> ***identity element***
>
> that number which, when combined with another according to a certain operation, does not change the other's value

For example, the identity element for addition is 0, because when 0 is added to any number, the value of that number does not change.

This is enough new terms for now; let's practice what we have learned, then proceed to learning a little about the most basic four functions of arithmetic.

Exercises 4.1

1. State the meaning of the following in your own words, as best as you can. Answer
- **1(a)** commutativity Answer
- **1(b)** associativity Answer
- **1(c)** anticommutativity Answer
- **1(d)** function Answer

2. State the word which covers the listed meaning. Answer
- **2(a)** Reversing the operands produces the negative result. Answer
- **2(b)** An operation in which each input produces one, and only one, output. Answer
- **2(c)** The order of the operands doesn't matter. Answer
- **2(d)** The order of the operations doesn't matter. Answer

Chapter 5
The Four Functions

All of arithmetic is built upon the basic foundation of the *four functions*. These functions—*addition*, *subtraction*, *multiplication*, and *division*—are the building blocks of everything else in arithmetic; with a grounding in these *four functions*, you'll be able to do mathematics from kindergarten to calculus, from probability and statistics to rocket science. Little can be considered more important for numeracy than a thorough understanding of these vital functions.

5.1 Addition

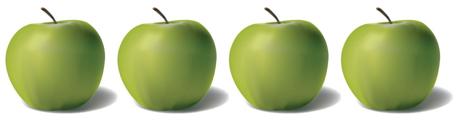
Addition is the most basic of the four functions, for good reason; it's the most frequently used of all of them. In fact, by one analysis, subtraction and multiplication are both simply forms of addition adapted for particular circumstances. So let's consider addition carefully, and develop our understanding of numbers as we go along.

5.1.1 Concept of Addition

Addition can be understood as simply joining groups of things together into a single group; it is equivalent to aggregation, combination, and any number of other words. As usual, the easiest way to grasp the concept is simply to do it; therefore, here we go.

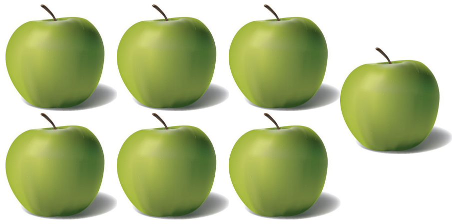
We have three apples:

We think this is enough for a group of three friends who are a little peckish on a hot summer day. But then four *more* friends show up; and *now* what are we to do? Rather than cutting up the apples (horrors!), we decide to fetch a few more. We need seven of them in total; we know this because we counted the number of friends we had with us (including ourself). We have on hand four more apples:



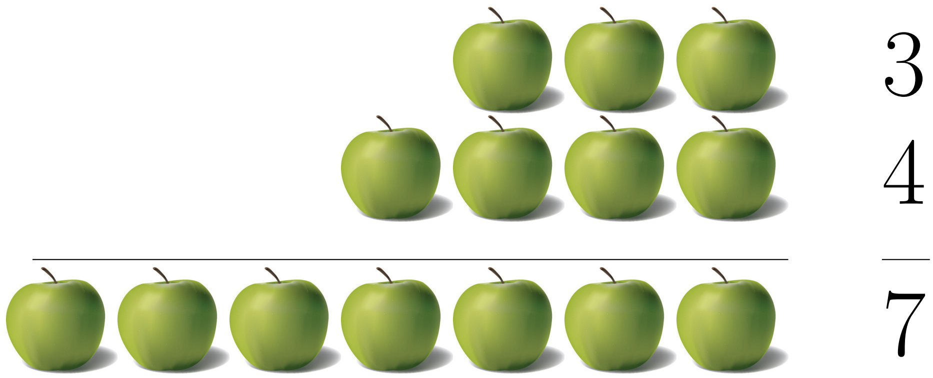
Do we have enough?

The easiest way to deal with this is to simply throw them all together and count them, and doubtlessly for many ages that is precisely what people did:



And we can easily simply count that we have seven (enough).

With large quantities of apples, this gets rather cumbersome; but we know about numbers! Instead of counting, we can just use numbers to short-circuit an otherwise laborious process.



The 3 and 4 here are the *addends*, an important enough word to require an explicit definition:

> ***addend***
>
> one of the operators in an addition operation; or, in other words, one of the numbers which will be added together

We then taken these addends (here there are two; there can be many more, if the situation requires it) and add them together to produce a *sum*:

> ***sum***
>
> the result of an addition operation; the total of the addends

And that's how we formulate an addition problem. Typically, the addends are placed on top of one another, with a line between them and the sum, and an addition sign, "+", to the left of the last addend:

$$\begin{array}{r} 3 \\ +\,4 \\ \hline 7 \end{array}$$

The addition sign is pronounced "plus" and is often referred to as a "plus sign"; the line between the addends and the sum is pronounced "equals." So the equation above should be read as "three plus four equals seven."

Note that addition is both *commutative* and *associative*; we can put the numbers in any order and the operations in any order and still get the same answer. In other words, the above equation $3 + 4 = 7$ is identical to the following:

$$\begin{array}{r} 4 \\ +\,3 \\ \hline 7 \end{array}$$

You can put the problem together however you like without having to worry about getting the wrong answer.

The *identity element* for addition is 0; you can add 0 to any number and wind up with the same number.

5.1.2 Addition Tables

Memorization has gotten a bad reputation among many educators in the last few dozen years, but it still serves its purpose, especially in basic mathematics. Sometimes the easiest way to start actually doing something is to simply memorize the most common use cases of doing it; and now that we have the basic concept of addition down, this may be a good time to memorize what we commonly call the *addition table*, which can be found below.

In the top and leftmost lines are the *addends*; where they meet in the table are the *sums*. To find the answer of $3 + ↊$, then, we need simply find three in one of these columns (in green), ↊ in the other (in blue), and find where the lines join; this will be the answer.

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	2	3	4	5	6	7	8	9	↊	↋	10	11
2	3	4	5	6	7	8	9	↊	↋	10	11	12
3	4	5	6	7	8	9	↊	↋	10	11	12	13
4	5	6	7	8	9	↊	↋	10	11	12	13	14
5	6	7	8	9	↊	↋	10	11	12	13	14	15
6	7	8	9	↊	↋	10	11	12	13	14	15	16
7	8	9	↊	↋	10	11	12	13	14	15	16	17
8	9	↊	↋	10	11	12	13	14	15	16	17	18
9	↊	↋	10	11	12	13	14	15	16	17	18	19
↊	↋	10	11	12	13	14	15	16	17	18	19	1↊
↋	10	11	12	13	14	15	16	17	18	19	1↊	1↋
10	11	12	13	14	15	16	17	18	19	1↊	1↋	20

As long as we stay within single digits, this table will suffice to answer all addition problems. Of course, we will not long remain within single digits; but it would do well for us to get some practice here before we proceed.

Exercises 5.1

1. 3 + ↊ = Answer	**2.** 6 + 5 = Answer	**3.** 2 + 2 = Answer	**4.** 3 + 6 = Answer
5. 6 + 3 = Answer	**6.** ↋ + 4 = Answer	**7.** 4 + ↊ = Answer	**8.** ↊ + 4 = Answer
9. 2 + 4 + 5 = Answer	**↊.** 5 + 2 + 4 = Answer	**↋.** 4 + 5 + 2 = Answer	**10.** 6 + 1 + 4 = Answer

5.1.3 Simple Addition

We can even add quite complex numbers using our addition tables this way. Let us take a simple example:

$$\begin{array}{r} 7\;6\;9\;2 \\ +\;4\;3\;1\;8 \\ \hline \end{array}$$

We simply add each digit in turn; that is, we add up the ones, and make that the ones digit of the answer:

$$\begin{array}{r} 7\;6\;9\;2 \\ +\;4\;3\;1\;8 \\ \hline \text{↊} \end{array}$$

Then the unqua digits, and make that the unqua digit of the answer:

$$\begin{array}{r} 7\;6\;9\;2 \\ +\;4\;3\;1\;8 \\ \hline \text{↊}\;\text{↊} \end{array}$$

Then the biqua digits, making that the biqua digit of the answer:

$$\begin{array}{r} 7\;6\;9\;2 \\ +\;4\;3\;1\;8 \\ \hline 9\;\text{↊}\;\text{↊} \end{array}$$

And finally, the triqua digits, making that the triqua digit of the answer:

$$\begin{array}{r} 7\ 6\ 9\ 2 \\ +\quad 4\ 3\ 1\ 8 \\ \hline \text{↋}\ 9\ \text{↊}\ \text{↊} \end{array}$$

And we have the answer to a four-digit addition problem using only our simple addition tables, with no complex jiggering required. This algorithm is quite powerful; as long as we keep our digits lined up, we can easily solve many addition problems with it.

5.1.4 Carrying

But our numbers don't always stay obediently within single digits; often they add up to more than 10. So what do we do in this case? We *carry*.

The important thing to remember here is simply what we learned long ago, in Section 1.3 (starting on page 5): each place doesn't mean just itself, but itself times 10 of the place to its right. This means we can simply *carry* the second digit of these larger numbers over to the next place and add it there.

To demonstrate:

$$\begin{array}{r} 7\ 9 \\ +2\ 7 \\ \hline \end{array}$$

We add the ones digits of the addends together as usual here: $9 + 7 = 14$. Notice that the sum of the ones digits, though, has an unqua digit (1). So we put the ones digits on the bottom, making it the ones digit of the total sum, and we *carry* the unqua digit to the next place to the left:

$$\begin{array}{r} {}^{1}\ \ \\ 7\ 9 \\ +2\ 7 \\ \hline 4 \end{array}$$

The 1, depicted in red, is the ones digit from the answer of $9 + 7 = 14$. Now, to get the unqua digit, we add 7 and 2, *but also the one which we carried over*:

$$\begin{array}{r} {}^{1}\ \ \\ 7\ 9 \\ +2\ 7 \\ \hline \text{↊}\ 4 \end{array}$$

Note that $7+2+1=↊$. Note also that it will not always be a one that we carry over; imagine, for example, that we are adding many numbers together:

$$\begin{array}{ccccc} & 7 & 9 & 2 & 7 \\ & 4 & ↋ & 6 & 8 \\ & 3 & 9 & 7 & 4 \\ + & ↊ & 9 & 4 & 3 \\ \hline \end{array}$$

First, we add all four of the ones digits of the addends together, $7+8+4+3$:

$$\begin{array}{ccccc} & 7 & 9 & 2 & 7 \\ & 4 & ↋ & 6 & 8 \\ & 3 & 9 & 7 & 4 \\ + & ↊ & 9 & 4 & 3 \\ \hline \end{array}$$

The answer to that $(7+8+4+3)$ is 1↊; we we put down our ↊ for the ones digit of the sum and then *carry* the unqua digit, 1:

$$\begin{array}{ccccc} & & & 1 & \\ & 7 & 9 & 2 & 7 \\ & 4 & ↋ & 6 & 8 \\ & 3 & 9 & 7 & 4 \\ + & ↊ & 9 & 4 & 3 \\ \hline & & & & ↊ \end{array}$$

Now, we add the unqua digits of the addends, giving us the problem $2+6+7+4$; but we also add the 1 which we carried over from the ones digits. The answer to that is 18:

$$\begin{array}{ccccc} & & 1 & 1 & \\ & 7 & 9 & 2 & 7 \\ & 4 & ↋ & 6 & 8 \\ & 3 & 9 & 7 & 4 \\ + & ↊ & 9 & 4 & 3 \\ \hline & & & 8 & ↊ \end{array}$$

Now we do the biqua place; the problem here is $9 + ↋ + 9 + 9$, plus the 1 which we carried over from the unqua place. $1 + 9 + ↋ + 9 + 9 = 33$; so this time, while we place the ones digit of the result here for the biqua of our sum, we now have to carry a 3, not a 1:

$$
\begin{array}{rcccc}
 & ^{3} & ^{1} & ^{1} & \\
 & 7 & 9 & 2 & 7 \\
 & 4 & ↋ & 6 & 8 \\
 & 3 & 9 & 7 & 4 \\
+ & ↊ & 9 & 4 & 3 \\
\hline
 & & 3 & 8 & ↊
\end{array}
$$

Now we have only one more digit, the triqua place, to add up. Our problem is $7+4+3+↊$, plus the 3 which we carried over. $3 + 7 + 4 + 3 + ↊ = 23$. But what do we do with our unqua digit here? There's nowhere to carry it!

Good; we can just put the sum of this smaller problem without carrying it at all:

$$
\begin{array}{rcccc}
 & ^{3} & ^{1} & ^{1} & \\
 & 7 & 9 & 2 & 7 \\
 & 4 & ↋ & 6 & 8 \\
 & 3 & 9 & 7 & 4 \\
+ & ↊ & 9 & 4 & 3 \\
\hline
2 & 3 & 3 & 8 & ↊
\end{array}
$$

It's worth noting that, because of place notation, the exact same algorithm works with fractionals, too: simply carry over the additional digits at each place:

$$
\begin{array}{rcccccc}
 & & & ^{1} & & ^{1} & \\
 & 4 & 2 & 3; & ↊ & 4 & 5 \\
+ & & 1 & 6; & 7 & 4 & 9 \\
\hline
 & 4 & 3 & ↊; & 5 & 9 & 2
\end{array}
$$

Notice that $↊ + 7 = 15$, and we carry the 1 right across the *radix point*, or *dit*, as if it's not even there. The thing to remember here is to simply line up the *dits*, even if the number of places is not the same. If the number of places are different, simply fill in the extra places with zeroes.

For example, let's add ↋62;4 and 493;158:

$$
\begin{array}{rcccccc}
 & & ^{1} & & & & \\
 & & ↋ & 6 & 2; & 4 & 0 & 0 \\
+ & & 4 & 9 & 3; & 1 & 5 & 8 \\
\hline
 & 1 & 4 & 3 & 5; & 5 & 5 & 8
\end{array}
$$

Of course, addition is *commutative*, so if you'd rather, you can reverse the numbers:

$$\begin{array}{r} {}^{1} \\ 493;158 \\ +\ \text{↋}62;400 \\ \hline 1435;558 \end{array}$$

The same technique works if you're adding integers with different numbers of digits; for example, 47 and 9↊71:

$$\begin{array}{r} 0047 \\ +\,9\text{↊}71 \\ \hline 9\text{↊}\text{↋}8 \end{array}$$

Or, the same problem:

$$\begin{array}{r} 9\text{↊}71 \\ +\,0047 \\ \hline 9\text{↊}\text{↋}8 \end{array}$$

Remember that, if there's no fractional part, then you just line the numbers up on the *right*; if there is a fractional part, you line the numbers up on the *dit*.

You now know enough to add any positive numbers in any configuration. (Adding negative numbers is really *subtraction*, or rather subtraction is really adding negative numbers, which we will discuss in Section 5.2.5 starting on page 4↋.) Let's have some practice, then, to help us master the algorithm we've learned.

Exercises 5.2

11. $\begin{array}{r} 4321 \\ +\,\text{↋↊}98 \\ \hline \end{array}$ Answer

12. $\begin{array}{r} 1234 \\ +\,89\text{↊↋} \\ \hline \end{array}$ Answer

13. $\begin{array}{r} 24 \\ +\,52954 \\ \hline \end{array}$ Answer

14. $\begin{array}{r} 1234 \\ 5678 \\ +\ 9\text{↊↋} \\ \hline \end{array}$ Answer

15. $\begin{array}{r} \text{↋↊}98 \\ 7654 \\ +\ 321 \\ \hline \end{array}$ Answer

16. $\begin{array}{r} 4\text{↋}692 \\ 34\text{↊}3 \\ +\,965\text{↋}4 \\ \hline \end{array}$ Answer

17. $\begin{array}{r} 7\text{↋}2 \\ 19325 \\ +\,\text{↊}4891 \\ \hline \end{array}$ Answer

18. $\begin{array}{r} 324 \\ 92\text{↊} \\ 16 \\ \text{↋↊↋} \\ 4\text{↊}6 \\ +\,453 \\ \hline \end{array}$ Answer

19. $\begin{array}{r} 2217 \\ 982\text{↊} \\ 6 \\ 47\text{↊↋} \\ \text{↊}6 \\ +\,5553 \\ \hline \end{array}$ Answer

1↊.
```
     1 2;3 4
     5 6;7 8
 +   9 ↊;↋ 1
 -----------
```
Answer

1↋.
```
     ↋ ↊;9 8
     7 6;5 4
 +   3 2;1 ↋
 -----------
```
Answer

20.
```
   ↋ ↊;9
   8 7;6
 + 5 4;3
 -------
```
Answer

21.
```
       5 3 2;6 8 9
     4 ↊ 9 4;3 2
 +         ↋ 9;5 6 9 8
 ---------------------
```
Answer

22.
```
        ↊ 9 3 1;7 4
        4 ↋ 8 4;3 2 6 9
            2 4 9
 +  6 2 9 8 ↊;3 8 5 9
 ---------------------
```
Answer

23. You are working in a snack booth at an amusement park, selling ice cream cones. Cones are \$2;60 for adults and \$2;00 for children, or \$5;00 for every three children. (Bulk discount, of course.) Answer

- **23(a)** A party of two adults and three children comes to the booth and each gets a cone. How much money must you collect? Answer
- **23(b)** A party of three adults and five children arrives; each gets a cone. How much money must you collect? Answer
- **23(c)** A party of two children and four adults arrives; each gets a cone. How much money must you collect? Answer

24. You are keeping score at a swim meet in which the lowest cumulative time (that is, the lowest sum of all times) over three races will be the final winner. You have to compute the scores for three separate races. Answer

- **24(a)** The first racers score the following times. Racer #1: 43;98, 44;23, 48;↊1. Racer #2: 42;77, 45;83, 4↊;11. What are the times? Who is the winner? Answer
- **24(b)** The second racers score the following times. Racer #3: 45;98, 44;3↊, 46;33. Racer #4: 49;54; 47;32; 44;39. What are the times? Who is the winner? Answer
- **24(c)** The third racers scores the following times: Racer #5: 42;72, 43;54; 45;↋3. Racer #6: 43;64, 43;74, 43;58. What are the times? Who is the winner? Answer
- **24(d)** Who is the overall winner? Answer

5.2 Subtraction

SUBTRACTION IS THE ACT OF TAKING AWAY rather than adding to. This definition seems trivial, but it is brief and accurate. *Subtraction* is *anticommutative* (that is, subtracting numbers in a different order gives a different answer) and it is not *associative* (that is, if you have more than one subtraction to do, doing them in a different order will produce a different answer).

Because of this anticommutativity, the numbers which we plug into a subtraction problem have different names (as opposed to addition, where both the operands are called addends). As we've discussed, subtraction is taking one number away from another number; the number we are taking away from is called the *minuend*.

minuend

the *first* input into the subtraction function (that is, the first number in a subtraction problem); the number from which we are taking away

The number we are taking away from the minuend is called the *subtrahend*:

subtrahend

the *second* input into the subtraction function (that is, the second number in a subtraction problem); the number which we are taking away from the minuend

And finally, after we have taken the subtrahend away from the minuend, we have learned the *difference* between the two numbers.

difference

the answer to a subtraction problem; the distance in numbers from the subtrahend to the minuend, or vice-versa

The name *difference* is not just jargon; when we subtract, we are literally finding the difference between two numbers. If you use the word "difference" when you're trying to find the answer to some problem, you are likely dealing with a subtraction problem.

Furthermore, problems of *comparison* usually require you to know the difference between two numbers. If a problem asks you how much more or how much less something is, you're likely looking at a subtraction problem.

Because of the fundamental difference between the minuend and the subtrahend, subtraction is not as simple a matter as addition is. However, there is still nothing to fear, and the rules can be learned very quickly.

One might think that, as with addition, the *identity element* in subtraction is 0; any number minus 0 equals the number, with no change in value. However, because subtraction is anticommutative, the reverse is not true; that is, while any number minus 0 is the same number, 0 minus any number is not. So strictly speaking, subtraction does not have an identity element; instead, 0 is a *right-identity*, in that when placed on the *right* of a problem, the value of the other operand does not change.

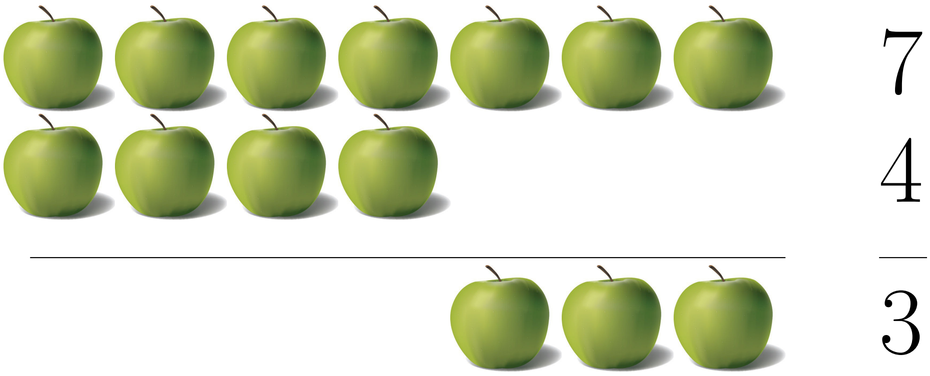
5.2.1 The Concept of Subtraction

In our section on addition,[1] we illustrated the concept by combining three apples and four apples (the addends) to make seven apples (the sum). Written in numbers, that was $3 + 4 = 7$. Let's now do the opposite; for in a sense, subtraction is simply the opposite of addition.

[1] *See supra*, Section 5.1, at 37.

You're having a picnic and you don't know how many people will be coming along. You have seven apples, and one by one four people each come and take one. How many do you have left?

You could count them, of course, and that works well enough for small numbers. But sometimes our numbers won't be small, and counting them simply won't be convenient. What if we're talking about *triquas* of people, and *triquas* of apples? Rather than trying to count, let's see how to find the difference without counting, so that we can handle any situation we come across.



In this way we are left with the problem which we write as follows:

$$\begin{array}{r} 7 \\ -\,4 \\ \hline 3 \end{array}$$

We voice this as "seven minus four equals three," or for young children, sometimes even "seven, take away four, is three." And it's easy to see that this is the opposite of addition; if $3 + 4 = 7$, then $7 - 4 = 3$.

Remember also that subtraction is *anticommutative*; changing the order of the numbers *does* change the answer, specifically changing the sign of the answer:

$$\begin{array}{r} 7 \\ -\,4 \\ \hline 3 \end{array} \qquad\qquad \begin{array}{r} 4 \\ -\,7 \\ \hline -\,3 \end{array}$$

As you can see, it's important that, when we're dealing with real, physical things that we're counting, we always keep the *larger* number as the minuend and the *smaller* number as the subtrahend; that is, that the bigger number be on the top and the smaller on the bottom. This is important if we want to keep our answer *positive*; that is, greater than zero.

If the larger number is being subtracted from the smaller number, the difference will be *negative*. Negative numbers are useful in many ways, even though we can never have less than zero of a certain thing; but let's get a good grasp on positive subtraction first, and talk about negative numbers in Section 5.2.4.

5.2.2 Simple Subtraction

Simple subtraction of single-digit numbers works exactly as addition does, except that you start with the *sum* and end with one of the *addends*. Let's look at two arithmetic problems beside one another, so we can more clearly see what we mean.

$$\begin{array}{r} 6 \\ +\,3 \\ \hline 9 \end{array} \qquad\qquad \begin{array}{r} 9 \\ -\,3 \\ \hline 6 \end{array}$$

Because of this, we can simply use the addition table quite similarly to the way we used it for addition. Whereas with addition the sum is where the rows for the addends meet, with subtraction we find the subtrahend; follow it down until we reach the minuend; and then follow it to the leftmost column to learn the difference.

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	2	3	4	5	6	7	8	9	↊	↋	10	11
2	3	4	5	6	7	8	9	↊	↋	10	11	12
3	4	5	6	7	8	9	↊	↋	10	11	12	13
4	5	6	7	8	9	↊	↋	10	11	12	13	14
5	6	7	8	9	↊	↋	10	11	12	13	14	15
6	7	8	9	↊	↋	10	11	12	13	14	15	16
7	8	9	↊	↋	10	11	12	13	14	15	16	17
8	9	↊	↋	10	11	12	13	14	15	16	17	18
9	↊	↋	10	11	12	13	14	15	16	17	18	19
↊	↋	10	11	12	13	14	15	16	17	18	19	1↊
↋	10	11	12	13	14	15	16	17	18	19	1↊	1↋
10	11	12	13	14	15	16	17	18	19	1↊	1↋	20

Of course, these problems are extremely simple, and without much practice the answers will be easily memorized. What do we do when we have larger numbers to deal with? In addition, we learned about *carrying*[2]; with subtraction, we must *borrow*.

5.2.3 Borrowing

When we add numbers together, we add the numbers one digit at a time; if we go over 10 in any digit, we carry the number of twelves to the next place. When we're finding differences, we often need to *borrow* a 10 from the next place over, in order to get enough to subtract from.

$$\begin{array}{r} 4\;↊\;9 \\ -\,6\;8\;3 \\ \hline \end{array}$$

[2] *See supra*, Section 5.1.4, at 3↋.

The first thing to do here is note that our minuend is larger than our subtrahend; otherwise, our algorithm for subtraction won't work. So let's switch them around to make sure that the minuend is larger:

$$\begin{array}{rccc} & 6 & 8 & 3 \\ - & 4 & \text{↊} & 9 \\ \hline \end{array}$$

Now we begin by subtracting the first column; as always, we do this just one digit at a time:

$$\begin{array}{rccc} & 6 & 8 & \color{red}{3} \\ - & 4 & \text{↊} & \color{red}{9} \\ \hline \end{array}$$

But as we can see, $3 - 9$ doesn't work; it's less than zero. So we need to *borrow* some number from the next place over, to keep us over zero.

Remember, though, that the next place over is really *10 times* the place we're working on; so we're not borrowing 1, we're borrowing 10. So subtract one from the next place over, and add 10 to the place we're working on:

$$\begin{array}{rccc} & & {\scriptstyle 7} & {\scriptstyle 13} \\ & 6 & 8 & \color{red}{3} \\ - & 4 & \text{↊} & \color{red}{9} \\ \hline \end{array}$$

When we borrow this way, it's the borrowed number that we worry about, not the original. So the subtraction for the first column is $13 - 9$, which equals 6. Moving on, we'll notice that we again have a problem; once again, the upper digit is smaller than the lower. So we need to borrow again; this time, borrowing 10 from the next digit to the left:

$$\begin{array}{rccc} & {\scriptstyle 5} & {\scriptstyle 17} & {\scriptstyle 13} \\ & 6 & \color{red}{8} & 3 \\ - & 4 & \color{red}{\text{↊}} & 9 \\ \hline & & & 6 \end{array}$$

So now we perform the subtraction for the second column, $17 - \text{↊}$, which equals 9. And then we move on to our next column, which doesn't need any borrowing because the top digit is larger than the lower. $5 - 4 = 1$.

$$\begin{array}{rccc} & \scriptstyle 5 & \scriptstyle 17 & \scriptstyle 13 \\ & 6 & 8 & 3 \\ - & 4 & \text{↊} & 9 \\ \hline & 1 & 9 & 6 \end{array}$$

And there is our answer. If we simply go one digit at a time, and remember that we must borrow from the next digit if the top digit is smaller than the bottom, then we really can't go wrong.

Note also that we may have to borrow across more than one digit before we can begin to subtract. In those cases, we must borrow from the farthest digit, then borrow from the new total, and so forth.

More simply, let us take the simple problem below. We'll note that it's already arranged so that the minuend is larger than the subtrahend, so we need no further work there.

$$\begin{array}{rcccccc} & 9 & 3 & 2 & 4; & 5 & 6 \\ - & 3 & 8 & \text{↊} & 4; & 8 & 7 \\ \hline \end{array}$$

Now we start with the first digit and note that the upper one is smaller than the lower one, so we must *borrow* to do the problem. However, when we try to borrow from the next place, we notice that it too has the same problem. So we must continue on until we find a place in which the upper digit is larger than the smaller one:

$$\begin{array}{rcccccc} & 9 & 3 & 2 & 4; & 5 & 6 \\ - & 3 & 8 & \text{↊} & 4; & 8 & 7 \\ \hline \end{array}$$

(Note that we will *always* find one eventually, because the minuend is larger than the subtrahend, so at least one digit will be appropriate for us to borrow from.)

Then we carry it over from that place to the first place we needed extra numbers, borrowing a 10 from each place as we go:

$$\begin{array}{rcccccc} & \scriptstyle 8 & \scriptstyle \cancel{13}12 & \scriptstyle \cancel{12}11 & \scriptstyle \cancel{14}13 & \scriptstyle \cancel{15}14 & \scriptstyle 16 \\ & 9 & 3 & 2 & 4; & 5 & 6 \\ - & 3 & 8 & \text{↊} & 4; & 8 & 7 \\ \hline \end{array}$$

In words, we are here borrowing one from the nine and making it eight, then adding 10 to 3 to make it 13; then we borrow one from 13, making it 12, and moving that 10 to the 2, making it 12; then we borrow one from the 12, to make it 11, and take that 10 to the

4, to make it 14; then we borrow one from the 14 to make it 13, and take it to the 5 to make it 15; then, finally, we borrow one from the 15 to make it 14, then give that it to the 6 to make it 16. It's not as complex as it sounds; it's just moving one from one place to another, as simple as that done in our last example.

Then, we simply solve, subtracting one place at a time, using the borrowed values (the smaller numbers on the top) rather than the originals:

$$\begin{array}{rcccccc} & {\scriptstyle 8} & {\scriptstyle \cancel{13}12} & {\scriptstyle \cancel{12}11} & {\scriptstyle \cancel{14}13} & {\scriptstyle \cancel{15}14} & {\scriptstyle 16} \\ & 9 & 3 & 2 & 4; & 5 & 6 \\ - & 3 & 8 & ↊ & 4; & 8 & 7 \\ \hline & 5 & 6 & 3 & ↋; & 8 & ↋ \end{array}$$

As noted in the problem above, the presence or absence of a fractional part makes no difference. Like our algorithm for addition, this algorithm works no matter what the numbers are. As for addition, we simply line up the *dits* (;) and borrow across the dit (or *radix point*) if necessary, as if it were not even there.

It is important, though, that we line up the dits, even if the number of digits after the dit are not the same. We typically leave these blank, but treat them as if there were zeroes there, borrowing to fill them in as needed:

$$\begin{array}{rccccccc} & & & & {\scriptstyle 7} & {\scriptstyle 15} & & \\ & ↋ & ↊ & 9 & 8; & 5 & 8 & 7 \\ - & 7 & 6 & 5 & 4; & 6 & & \\ \hline & 4 & 4 & 4 & 3; & ↋ & 8 & 7 \end{array} \qquad \begin{array}{rcccccccc} & & & {\scriptstyle 7} & {\scriptstyle \cancel{12}11} & {\scriptstyle 13} & {\scriptstyle ↊} & {\scriptstyle \cancel{10}↋} & {\scriptstyle 10} \\ & 4 & 5 & 8; & 2 & 3 & ↋ & & \\ - & & 2 & 5; & 4 & 5 & 9 & 8 & 2 \\ \hline & 4 & 3 & 2; & 9 & ↊ & 1 & 3 & ↊ \end{array}$$

We can fill in the zeroes here, as zeroes at either the far left or the far right side of a number don't change its value; we've chosen to leave them out. (That is, our second problem could be written as 458;23↋00 − 025;45982, if it helps you visualize the borrowing better.) Note also that you can even *borrow* to one of these zeroes, and then borrow from it. Remember, you're simply taking 10 from the place immediately to the left and adding it to your figure; if your figure is 0, then you add 10 to it, making it simply 10. There is nothing special about this case.

And now we know the algorithm which allows us to subtract; let's try a few examples.

Exercises 5.3

25. $\begin{array}{r} ↋\,↊\,9\,8 \\ -\,7\,6\,5\,4 \\ \hline \end{array}$ Answer

26. $\begin{array}{r} ↋\,4\,↊\,5 \\ -\,9\,6\,8\,7 \\ \hline \end{array}$ Answer

27. $\begin{array}{r} 5\,2\,9\,5\,4 \\ -\,4\,5\,9\,2\,5 \\ \hline \end{array}$ Answer

28. $\begin{array}{rl} & 4\,5\,2\,9;↋\,3 \\ - & \;\;3\,8\,↊;4\,4\,4 \\ \hline \end{array}$ Answer

29. $\begin{array}{rl} & 9\,↋\,4\,9;2\,3\,8 \\ - & 5\,↋\,4\,3;3\,6\,8\,3 \\ \hline \end{array}$ Answer

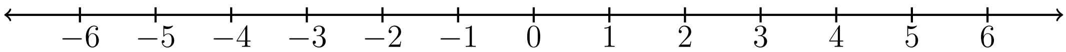
2↊. You have \$45 and your friend has \$132. How much more cash does your friend have than you? (Hint: this is asking the *difference* between your holdings and your friend's.) Answer

2↋. You are judging bids made by two companies to perform a project for your company. One company bid \$45↋3;↋↋ for the project and will take twelve days to complete it, while the other bid \$49↊↊;↋↋ and will take one week. Answer

- **2↋(a)** How much lower is the lower bid than the higher? Answer
- **2↋(b)** How much longer will the lower bid take than the higher? Answer
- **2↋(c)** Which company should get the project? Answer

5.2.4 Negative Numbers

In Section 1.4,[3] we saw that it's possible to count *backwards* from zero; when we do this, we have *negative numbers*, and we pronounce them the same as we do positive numbers, but prefix them with the word "negative" and write them with a *minus sign*, −. To refresh our memory, here is a number line showing how it works:



We simply count down to zero, then begin counting up again, but with "negative" in front of it; e.g., −1, −2, −3, −4 . . ..

We bring negative numbers up again here because subtraction first brings them to the fore, and also because in a way subtraction can be thought of as simply addition of negative numbers.[4] Why does subtraction bring negative numbers to the fore? Let's see what the anticommutativity of subtraction does when we try to rearrange the digits in a simple problem.

$$\begin{array}{r} 1\ 0 \\ -\quad 7 \\ \hline 5 \end{array} \qquad\qquad \begin{array}{r} 7 \\ -1\ 0 \\ \hline -\ 5 \end{array}$$

We see that this is an easy problem; because subtraction is *anticommutative*, reversing the order means that we get the same answer, *but negative*. Since our algorithm for subtraction only works when the minuend is larger than the subtrahend, all we need to do is reverse the numbers, then *negate* the answer.

In other words, if we are presented with the problem 7 − 10, to find the answer we should subtract 10 − 7, get the answer (5), and then negate it (−5). This is the simplest way to deal with this situation.

Sometimes, however, we run into problems with negative numbers already involved. For example, consider a business which is "in the red"; that is, which owes more money than it has. Its finances are most easily described as *negative*. To determine anything about its finances, then, we will have to deal with *negative numbers*.

[3] *See supra* at 8. [4] *See infra*, Section 5.2.5, at 4↋.

5.2.5 Adding and Subtracting with Negatives

This section involves both addition and subtraction with negative numbers because the rules are the same. More accurately, a negative number can turn a subtraction problem into an addition problem, or an addition problem into a subtraction problem. Here's how.

Let's consider a starving artist who's overdrawn his checking account; the bank says that his balance is −\$458;32. Aside from learning a lesson about more prudential spending, our artist realizes that he needs \$260 in order to pay the rent for his studio apartment in a few days. How much money does he need to make before he gets that \$260 in his checking account?

Fundamentally, of course, this is a question of comparing, and therefore one of *difference*. This means it's a subtraction problem, which we can write like this:

$$\begin{array}{r} \text{-}\ 4\ 5\ 8;3\ 2 \\ -\qquad 2\ 6\ 0 \\ \hline \end{array}$$

But if we do this problem as normal, without taking into account the signs, we will get \$1Ɛ8;32, which obviously isn't the right answer; that won't even get him above zero! No, somehow we must acknowledge the sign if we have any hope of figuring this out.

Firstly, let's remember that a negative number is lower than zero; that means that the problem we've written above has a minuend which is smaller than its subtrahend. That doesn't work; so let's rearrange that to be correct:

$$\begin{array}{r} 2\ 6\ 0 \\ -\quad -4\ 5\ 8;3\ 2 \\ \hline \end{array}$$

Now let's note that this puts two *minus signs* right next to one another, which doesn't look right. In fact, it isn't right; two minus signs equals a plus sign. That is, subtracting a negative is the same as adding a positive:

$$\begin{array}{r} 2\ 6\ 0 \\ +\qquad 4\ 5\ 8;3\ 2 \\ \hline 6\ \text{Ɛ}\ 8;3\ 2 \end{array}$$

And the artist needs \$6Ɛ8;32 if he wants to pay his rent; hopefully he can sell some paintings before it comes due.

We could do this for every possible combination of positives and negatives, but there's no need; the principle is clear. It all boils down to two simple rules:

- Two like signs (either two plus signs or two minus signs) become a plus sign

- Two unlike signs (a plus and a minus or a minus and a plus) become a minus sign

Remember here that unmarked numbers are always positive, even if they don't have a plus sign in front of them.

Examples

1. $2 + -4$. This is obviously two unlike signs here, so the problem we need to do is $2 - 4$. Here, though, our minuend is smaller than our subtrahend, so we switch them around and solve: $4 - 2 = 2$ But subtraction is anticommutative, so we must negate the answer to get the real answer, -2.

2. $2 - -4$. Two like signs make a plus, so the problem we're doing here is $2 + 4 = 6$.

3. $-2 - 4$. Trickier here, because the signs don't actually come next to each other. Switch them around: $4 - -2$. Two like signs, which makes a positive. $4 + 2 = 6$. However, we switched around our operands, and subtraction is not commutative, so we negate that to get the real answer, -6.

4. $-2 + 4$. Just as before, switch them around to make $4 + -2$. Two unlike signs, so the real problem is $4 - 2 = 2$. Because, unlike in the last example, we started with addition, which is commutative, we do not need to negate the answer; 2 is correct.

5. $-2 + -4$. Two unlike signs here, leaving us $-2 - 4$. Then, switch them around to get the signs next to one another: $4 - -2$. Two like signs, so $4 + 2 = 6$. But remember that we switched the operands here when we had a *subtraction*, so we can't switch the order unless we negate the answer. So the true answer is -6.

6. $-2 - -4$. Two like signs make a plus, so $-2 + 4$. Switch them around: $4 + -2$. Two unlike signs, so we really have a subtraction: $4 - 2 = 2$. Notice that we never changed the order when we had a subtraction; if we had, we would have to negate the answer. Since we didn't, we don't, and 2 is correct.

Eventually, you will able to tell these things without formally going through all the steps (though be careful; a mistake can result in drastically wrong anwers).

Exercises 5.4

30. −4Ɛ2;78 + 28;495 Answer **31.** −4Ɛ2;78 − 28;495 Answer **32.** 2;975 − 3;45 Answer
33. −2;975 + −3;45 Answer **34.** −2;975 − −3;45 Answer **35.** 2;975 + −3;45 Answer

36. You are the accountant for a small pet shop, which is suffering some financial difficulties. Every month, the pet shop must pay \$1680 of rent; \$340 for electricity; \$90 for water; \$980 for food for the stock; and \$300 of protection money to the local mob. (It's a bad neighborhood.) Average income per month is \$2400. Answer

36(a) What are the business's total outlays each month? Answer

36(b) What is the budgetary shortfall per month? Answer

36(c) Assuming no savings carrying over from month to month, what is the business's bank account value after all expenditures and income? Answer

36(d) You estimate that an advertisement in a local trade paper could bring in an additional \$900 per month; however, the cost of that advertisement would be

$100 per month. Is it worth it? Answer

36(e) Your estimate was really that this ad would bring between $820 and $1030 in extra income. Calculate monthly balance assuming the lower and the higher threshhold. Answer

5.2.6 Checking Addition and Subtraction Results

We can use the fact that addition and subtraction are, at root, simply two aspects of the same thing in order to *check our answers*. For example, if we do a long or complex addition problem, we can do it again, backwards, as subtraction in order to see if our answer is correct. The *sum* in the addition problem simply becomes the *minuend* in the subtraction problem, like so:

```
    4 ↊ 9 8 3          1 0 2 ↋ ↊ 2
+   7 4 2 1 ↋      −     7 4 2 1 ↋
-------------      -------------
  1 0 2 ↋ ↊ 2          4 ↊ 9 8 3
```

In this case, we've selected the second *addend* from the first problem to serve as the *subtrahend* for the second; we could just as easily and effectively choose the first. The important thing is that the same three terms are present in each problem; if that occurs, we can be assured that our answer is correct.

Oftentimes, we won't be unsure about our answer; however, if we are, it's often very instructive to do the problem again backwards with the opposite operation. After some experience with arithmetic, the student will begin to develope an "eye" for when an answer is approximately correct; if that "eye" indicates that the answer is wrong, it's profitable to check. Furthermore, if the calculation is of particular importance, it is *always* profitable to check, in more than one way if possible; for example, by checking with both addends alternately as the subtrahend.

5.3 Multiplication

Multiplication is, put very simply, nothing but repeated addition. That is to say: when we multiply, we simply add repeatedly until we get an answer. *Multiplication* thus enables us to do many common problems much more quickly than simple addition would allow.

To illustrate, let us consider a party in which each guest brings one case (one case is 10 drinks) of beverages to contribute. There are seven guests. How many drinks do we have?

At first glance, this is a question of *totals*; that makes it look like an *addition* question. However, look at what we need to do to answer this question by adding:

$$\begin{array}{r} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ +\quad 10 \\ \hline 70 \end{array}$$

This is not a *difficult* addition problem, but it's certainly a long and complex one. Doing it with numbers other than 10 makes it even more complex.

With *multiplication*, though, we can drastically decrease the amount of work we have to do:

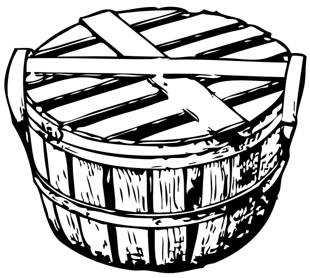
$$\begin{array}{r} 10 \\ \times\quad 7 \\ \hline 70 \end{array}$$

We pronounce this as *unqua times seven equals seven unqua*, and we can simply call it a day.

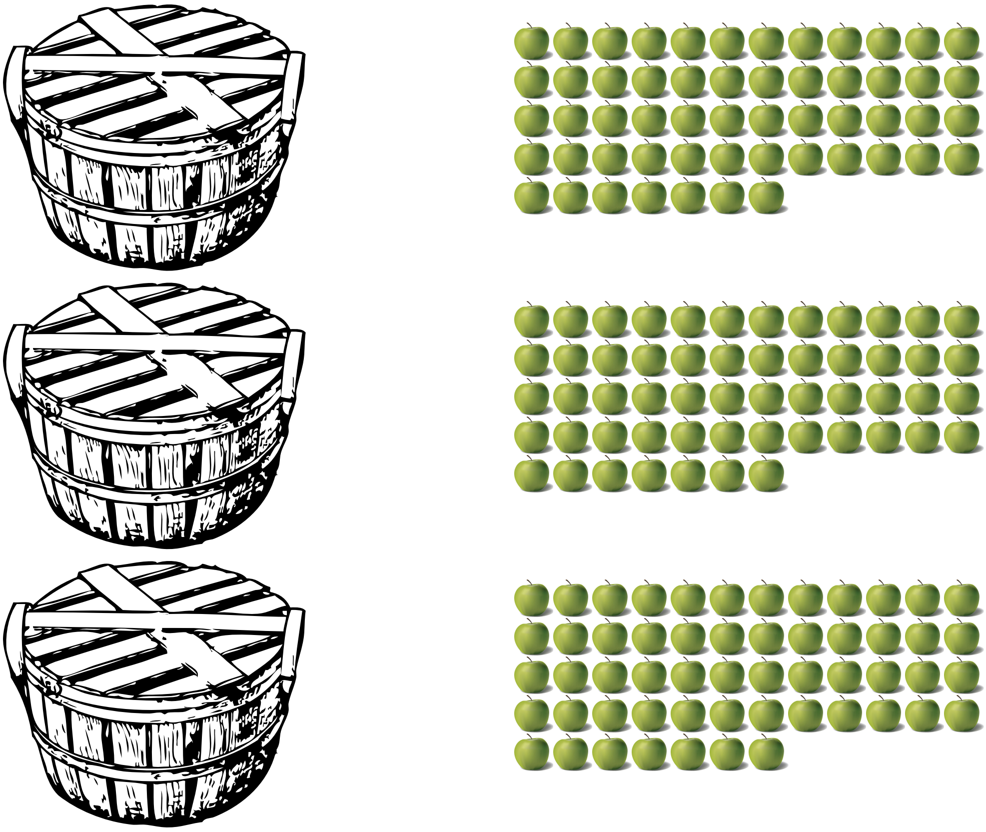
The multiplication sign is pronounced either *times* or *multiplied by*, and it can be written either as $\times$ or as $\cdot$ (a dot centered vertically on the line). Also, simple juxtaposition can indicate multiplication; that is, simply putting two things next to one another means that they are to be multiplied. Newton's famous Second Law of Motion, that force equals mass times acceleration, is written symbolically as $F = ma$; the m and the a beside one another mean that they are to be multiplied. Obviously, we can't do this with simple numbers, though, so in this text you'll mostly see $\times$ and $\cdot$.

We've used apples as examples for addition and subtraction; let's switch it up for multiplication and use *bushels* of apples. Since multiplication is repeated addition, we can determine the number of apples in a certain number of bushels with it.

Each bushel looks like this:



Each bushel contains 47 apples. (This is purely academic, and bears no relation to what an actual apple bushel might hold.) So how many apples do we have if we have *three* bushels? Calculating this by *addition* would look something like this:



$$\begin{array}{r} 47 \\ 47 \\ +\,47 \\ \hline \end{array}$$

Certainly doable, but inconvenient. So instead, we *multiply*: as above, with our basic 10×7 example above, we use a multiplication algorithm to add repeatedly in only one operation.

$$\begin{array}{r} 4\ 7 \\ \times\ \ \ 3 \\ \hline 1\ 1\ 9 \end{array}$$

If this seems suspiciously easy, go ahead and do the addition to verify that it really works:

$$\begin{array}{r} \overset{1}{4}\ 7 \\ 4\ 7 \\ +\,4\ 7 \\ \hline 1\ 1\ 9 \end{array} \qquad \begin{array}{r} 4\ 7 \\ \times\ \ \ 3 \\ \hline 1\ 1\ 9 \end{array}$$

Still, as simple as the multiplication problem is, the addition problem is pretty easy, as well. But consider trying to calculate how many apples are in, say, 543 bushels; addition is prohibitively labor-intensive, but with multiplication, it's still quite easy. Multiplication is an ideal solution to a large body of problems.

5.3.1 Multiplication Definitions

As in addition and subtraction, each part of a multiplication problem has different words associated with it. Multiplication is both *commutative* and *associative*; that is, the order in which two numbers are multiplied has no effect on the answer, and the order of operations in which several numbers are multiplied has no effect on the answer.

factor

one of the terms of a *multiplication* problem

Because multiplication is commutative, like with the *addends* of an addition problem we often don't bother distinguishing between the two terms of a problem; both the first and second numbers can therefore be referred to as *factors*.

However, when we do wish to distinguish, we use the terms *multiplicand* and *multiplier*.

multiplicand

the number to be multiplied in a multiplication problem; typically the first number

The multiplicand is the number which will be multiplied; in the example on page 53, it was the number of apples in each bushel, 47.

multiplier

the number of addends, when multiplication is considered as repeated addition; the number of times the multiplicand is to be added together

The *multiplier* is the number of times the multiplicand is to be added together; in the example on page 53, this was the number of bushels, 3.

Finally, we need to have a name for the result of a multiplication operation; fortunately, this one is easy to remember, as long as we remember it as the *product* of the multiplication.

product

the result of a multiplication operation

And now, having learned the basic words associated with multiplication, we can move on to actually learning the algorithms by which we do it.

5.3.2 Multiplication Table

Just as with the addition table,[5] often the best way to learn multiplication is by the memorization and use of multiplication tables. Here is the table for multiplication in dozenal.

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	1	2	3	4	5	6	7	8	9	↊	↋	10
2	2	4	6	8	↊	10	12	14	16	18	1↊	20
3	3	6	9	10	13	16	19	20	23	26	29	30
4	4	8	10	14	18	20	24	28	30	34	38	40
5	5	↊	13	18	21	26	2↋	34	39	42	47	50
6	6	10	16	20	26	30	36	40	46	50	56	60
7	7	12	19	24	2↋	36	41	48	53	5↊	65	70
8	8	14	20	28	34	40	48	54	60	68	74	80
9	9	16	23	30	39	46	53	60	69	76	83	90
↊	↊	18	26	34	42	50	5↊	68	76	84	92	↊0
↋	↋	1↊	29	38	47	56	65	74	83	92	↊1	↋0
10	10	20	30	40	50	60	70	80	90	↊0	↋0	100

It's easy to see patterns here in almost all the digits, the only exceptions being 5, 7, and ↊. For example, the 2 table proceeds 2, 4, 6, 8, ↊, 0 in its final digit; the 3 table proceeds 3, 6, 9, 0 in its final digit; the 4 table proceeds 4, 8, 0 in its final digit; the 6 table proceeds 6, 0 in its final digit. Conversely, the 8 table proceeds 8, 4, 0 in its final digit (the opposite of its half, 4), and the 9 table table proceeds 9, 6, 3, 0 in its final digit (the opposite of its third, 3).

It is easiest to commit this table to memory; fortunately, these many patterns make this fairly easy. It is considerably easier to memorize this table than that for decimal, for example, though it has more problems in it, because more of the tables have simple patterns. By contrast, for example, the decimal table has easy patterns only for 2 and 5.

As an example, let us use the table in the same way we learned to use our addition table. For example, let's find the answer for

$$7 \times 9$$

[5] See *supra*, Section 5.1.2, at 39.

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	1	2	3	4	5	6	7	8	9	↊	↋	10
2	2	4	6	8	↊	10	12	14	16	18	1↊	20
3	3	6	9	10	13	16	19	20	23	26	29	30
4	4	8	10	14	18	20	24	28	30	34	38	40
5	5	↊	13	18	21	26	2↋	34	39	42	47	50
6	6	10	16	20	26	30	36	40	46	50	56	60
7	7	12	19	24	2↋	36	41	48	53	5↊	65	70
8	8	14	20	28	34	40	48	54	60	68	74	80
9	9	16	23	30	39	46	53	60	69	76	83	90
↊	↊	18	26	34	42	50	5↊	68	76	84	92	↊0
↋	↋	1↊	29	38	47	56	65	74	83	92	↊1	↋0
10	10	20	30	40	50	60	70	80	90	↊0	↋0	100

This technique will allow you to multiply any single-digit number by any other single-digit number, plus 10.

Exercises 5.5

37. 3×8 Answer

38. 7×5 Answer

39. 8×5 Answer

3↊. $9 \times$ ↊ Answer

3↋. ↊ $\times 4$ Answer

40. 8×9 Answer

41. 6×7 Answer

42. 5×5 Answer

43. 7×7 Answer

44. 4×7 Answer

45. 7×4 Answer

46. 9×3 Answer

47. 9×9 Answer

48. ↊ $\times$ ↋ Answer

49. ↋ $\times 5$ Answer

4↊. 5×9 Answer

5.3.3 Multiplication Facts

From the multiplication table, we can derive some very interesting, and very useful, multiplication facts. They are so useful, in fact, that it is worth the time to memorize them now.

First, we can see that multiplication's identity element is 1; that is, any number multiplied by 1 will equal itself. That means that, when we're dealing with multiplying by 1, we can simply ignore that part of the equation entirely.

Second, any number multiplied by 0 equals 0. This does not appear on the table, but it's nevertheless true; we can "prove" it by thinking about what multiplication is. If we take zero bushels of apples, times 47 apples per bushel, then how many apples do we have? Clearly, none, or 0.

A corollary of this: because multiplication is *commutative* and *associative*, any string of multiplication problems with a 0 in it will, as a whole, equal 0. E.g., $5 \times 47 \times 0 \times 389$ equals 0, simply because there is a 0 present in it.

Finally, multiplying by numbers ending in 0 is often quite easy. Notice the bottom and rightmost row of the multiplication table: each number multiplied by 10 is itself followed by a 0. E.g., $4 \times 10 = 40$, ↊ $\times 10 =$ ↊0. This is, in fact, true for *all numbers.* $48398 \times 10 = 483980$.

It gets even better, too. Whenever multiplying a number by a 1 followed by any number of zeroes, *simply add the same number of zeroes to the first number*. That will be the correct answer. We've seen that $48398 \times 10 = 483980$; what, then, is 48398×100000? The answer is 4839800000; since the multiplier has five zeroes, the product will be the multiplicand plus five zeroes.

We can extend this even further; multiplication by a 1 followed only by zeroes is simply *moving the radix point*, or the "dit," to the right. The problems of this type we've seen so far all have their dit at the far right of the number, with no digits following them; for numbers like this, we typically don't write the dit at all. But it's there, and when we move the dit to the right, we need to add a zero to fill it in. Consider:

$$43 \times 100 = 43; \times 100 = 4300; = 4300$$

Now try to apply the same principle to a number which has a dit followed by numbers. Consider 43;58.

$$43;58$$

When we multiply an integer by 10, we simply add a 0; but that's the same as moving the dit to the right one place. So when we multiply 43;58 by 10, we do just that:

$$43;58 \times 10 = 435;8$$

And just as we add two zeroes to an integer when multiplying by 100, we move the dit *two* places to the right here:

$$43;58 \times 100 = 4358;$$

Now, of course, we have an integer, so when we move the dit further to the right, we add zeroes. Let's try multiplying by 10000:

$$43;58 \times 10000 = 435800;$$

We have four zeroes in 10000, so we move the dit four places to the right; two places puts the dit on the far right of the number, so for the next two, we add zeroes. Now, when the dit is at the very right of the number, we don't need to write it anymore, so we see that $43;58 \times 10000 = 435800$. This little trick turns what could be a very difficult problem into a trivially easy one.

Note that this only works with a 1 followed by zeroes. However, we can still use this fact to make other multiplication problems easier. For example, let's consider $43;58 \times 2000$. By dividing this into two simpler problems, we can make this much easier.

Normally, we'd have to engage in long multiplication[6] to solve this problem, which, with a four-digit multiplier, would be a long process. But multiplication is commutative, so we can divide the problem up however we want to make it easier. $2000 = 2 \times 1000$, as we just learned. This turns our problem into $43;58 \times 2 \times 1000$. So let's multiply by 2, then by 1000:

[6] *See infra*, Section 5.3.4, at 58.

$$43;58 \times 2000$$
$$43;58 \times (2 \times 1000)$$
$$43;58 \times 2 \times 1000$$
$$86;\mathcal{E}4 \times 1000$$
$$86\mathcal{E}40;$$
$$86\mathcal{E}40$$

This gives us 86Ɛ40, which is indeed the answer, once again turning a difficult problem into an easy one.

So, to sum up some of the easy-to-remember rules about multiplication:

- Any number multiplied by 0 is 0.
- Any number multiplied by 1 is itself.
- When multiplying by a 1 followed only by zeroes, the product is the multiplicand with its dit moved to the right by the number of zeroes in the multiplier. In other words, count the zeroes and move the dit to the right that many times.

Now, a few exercises to make sure we've internalized these important rules.

Exercises 5.6

4Ɛ. 7389×0 Answer **50.** 7389×1 Answer **51.** 7389×10 Answer **52.** $73;89 \times 10$ Answer **53.** $73;89 \times 1000$ Answer **54.** $0;7389 \times 1000$ Answer **55.** $0;07389 \times 1000$ Answer **56.** $0;07389 \times 0$ Answer

5.3.4 Long Multiplication

We learned an *algorithm*, or set of steps, for solving addition problems back in Section 5.1; we called this *carrying*. We have a similar algorithm for multiplication; however, the matter isn't quite so simple. First, we will look at the algorithm for multiplying by single digits; then, we will extend that to multiplying by numbers of several digits.

Multiplying by a single digit is quite similar to simple addition; just multiply each digit by the single digit in turn. Remember that multiplication is *commutative*; that is, we can multiply in any order without effecting the answer. Typically, however, we put the shorter number on the bottom.

Let's demonstrate some simple multiplication by a single digit.

$$\begin{array}{rccc} & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline \end{array}$$

First we multiply the first digit by the bottom digit; we can do this from memory or by reference to the multiplication table:

$$\begin{array}{rccc} & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline \end{array}$$

$3 \times 8 = 20$, so we place a 0 in under that column and we *carry* the 2, just as we do in our algorithm for addition:

$$\begin{array}{rccc} & & {\scriptstyle 2} & \\ & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline & & & 0 \end{array}$$

When doing addition, we'd now add the second column together. But in multiplication, we must *multiply every digit of the multiplicand by every digit of the multiplier*. So instead, we use the ↊ in the second place of the multiplicand, but multiply it by the 8 in the first (and only) place in the multiplier, *adding* the 2 which we carried:

$$\begin{array}{rccc} & & {\scriptstyle 2} & \\ & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline & & & 0 \end{array}$$

$↊ \times 8 = 68$, and $68 + 2 = 6↊$, so we put down the ↊ and carry the 6:

$$\begin{array}{rccc} & {\scriptstyle 6} & {\scriptstyle 2} & \\ & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline & & ↊ & 0 \end{array}$$

Then we multiply the 4 and the 8, adding in the 6 that we carried before:

$$\begin{array}{cccc} & {\scriptstyle 6} & {\scriptstyle 2} & \\ & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline & & ↊ & 0 \end{array}$$

$4 \times 8 = 28$, and $28 + 6 = 32$. Since this is the final digit, we don't have to carry anything to the next place, and instead simply place the 32 into the number:

$$\begin{array}{cccc} & {\scriptstyle 6} & {\scriptstyle 2} & \\ & 4 & ↊ & 3 \\ \times & & & 8 \\ \hline 3 & 2 & ↊ & 0 \end{array}$$

And we have our answer. This is not much more complicated than simple addition; but it is much easier than adding 4↊3 to itself 8 times.

Things do get considerably more complicated when we want to multiply by numbers of more than one digit. Let's say we have 83 bushels of apples, each of which contains 48 apples. How many apples do we have?

We already know how to write out the problem. Since we're talking about multiple groups, each of which contains an equal number, this is definitely a multiplication problem; so let's start by writing that out:

$$\begin{array}{ccc} & 8 & 3 \\ \times & 4 & 8 \\ \hline \end{array}$$

So far, so good. The idea behind this will be to multiply 83 by 8 (83×8), and then multiply 83 by 40 (83×40), and finally add the answers. This is really quite simple. We begin by calculating 83×8, paying no regard to the 4 in the multiplier. We needn't see the steps here, since we already know how to do this.

$$\begin{array}{cccc} & & 8 & 3 \\ \times & & 4 & 8 \\ \hline & 5 & 6 & 0 \end{array}$$

Now we calculate 83×40. We do this by simply calculating 83×4, *and then adding a zero to the right side*. This works because, as we saw with our multiplication table in Section 5.3.2,[7] the answer to any number multiplied by 10 is the number itself, plus a

[7] *See supra* at 55.

zero on the right. So $7 \times 10 = 70$; $↊ \times 10 = ↊0$, and so on. So 83×40 is 83×4, plus a zero on the right. (Remember that 40 is just a way of saying "*four dozen*.")

We can make this easier by just putting the zero on the right before we solve the simpler problem:

$$\begin{array}{rcccc} & & & 8 & 3 \\ \times & & & 4 & 8 \\ \hline & & 5 & 6 & 0 \\ + & & & & 0 \\ \hline \end{array}$$

Now we do the multiplication of 83×4, exactly as we would if that were the whole problem. You should probably cross out any carries from the first round, so you don't confuse yourself with them, and you can place the new digit's carries on top of them. That is, multiply $3 \times 4 = 10$, carry the 1 and place the 0; then multiply $8 \times 4 = 28$, add the 1 you carried earlier, and then place it in your sum:

$$\begin{array}{rcccc} & & & 8 & 3 \\ \times & & & 4 & 8 \\ \hline & & 5 & 6 & 0 \\ + & 2 & 9 & 0 & 0 \\ \hline \end{array}$$

$83 \times 4 = 290$; so we place the 290 there, plus the 0 that we put in before (since we're multiplying by the second digit), and then add them together to get our answer:

$$\begin{array}{rcccc} & & & 8 & 3 \\ \times & & & 4 & 8 \\ \hline & & 5 & 6 & 0 \\ + & 2 & 9 & 0 & 0 \\ \hline & 3 & 2 & 6 & 0 \end{array}$$

And there you have it. This can be extended just as far as we care to, with numbers as long as necessary. The important thing to remember is that, for each extra digit of the multiplier, we must add one zero to the right of the answer. A four-digit multiplication problem is demonstrated below, with the necessary added zeroes in blue for clarity.

```
        5 Ɛ 9 7 4
×         3 8 7 9
-----------------
      4 5 ↊ 2 6 0
    3 5 ↊ 7 3 4 0
  3 Ɛ ↊ 4 ↊ 8 0 0
+1 5 Ɛ 4 ↊ 0 0 0 0
-----------------
 1 ↊ 3 1 7 4 1 ↊ 0
```

Once again, when doing such problems by hand, it's often helpful to erase or cross out the carries after multiplying each digit, so that you don't become confused by the multitude of carries placed above your top number.

But note, in the problem above, that one zero is added for each digit of the number except for the first. Why?

Again, because of the way we write our numbers.[8] The first place represents *ones*, the second *dozens*, and so on. So when we multiply the multiplicand by the first digit of the multiplier, we're multiplying it by ones, and so we don't need to add any zeroes. But when we multiply the multiplicand by the *second* digit of the multiplier, we're multiplying it by *dozens*, and so we need to add a zero to reflect that. When we multiply the multiplicand by the *third* digit of the multiplier, we're multiplying it by *biquas*, and so we need to add two zeroes to reflect this. And so on.

Exercises 5.7

57. Ɛ↊98 × 4 Answer

58. 4↊5 × ↊ Answer

59. 34 × 8 Answer

5↊. 857Ɛ23 × 6 Answer

5Ɛ. ↊987 × 65 Answer

60. 73853 × 68Ɛ Answer

61. 88ƐƐ44 × 3399↊↊ Answer

62. 3979 / 45 / × 672 Answer

63. Ɛ↊987 / 6543 / × 210 Answer

64. 24 / 39Ɛ4 / × 9532 Answer

65. You are planning a wedding reception and have invited 164 guests. Your caterer has informed you that the meal you selected will cost $46 per person. How much will the meals cost you? Answer

66. You operate a very large and busy Chinese restaurant, and your two specialties are General Ts'o chicken and sweet and sour pork. You use a maximum of 14 gallons of

[8] *See supra*, Section 1.3, at 5.

General Ts'o sauce per day and a maximum of 13 gallons of sweet and sour sauce per day. Answer

66(a) Suppose you receive supply deliveries every three days. How much General Ts'o sauce should you order? How much sweet and sour sauce? Answer

66(b) You also need to order chicken and pork. For every gallon of sauce you need 13 pounds of meat. How much of each should you order? Answer

5.3.5 Multiplying Digital Fractions

We saw that for *addition* and *subtraction* we can multiply *digital fractions* simply by lining up the *radix points* (or "*dits*"). For multiplication, the process is only slightly more complicated.

Instead of lining up the dits, we line up the numbers without regard to the positions of the dits. For example:

$$\begin{array}{r} 345;21 \\ \times \quad 3;4↊9 \\ \hline \end{array}$$

Then we solve the problem as if the dits were not even there:

$$\begin{array}{rcccccccc}
 & & & & 1 & 1 & & & \\
 & & & & \cancel{1} & \cancel{1} & & & \\
 & & & & \cancel{3} & \cancel{4} & \cancel{1} & & \\
 & & & & & \cancel{3} & \cancel{1} & & \\
 & & & & 3 & 4 & 5; & 2 & 1 \\
\times & & & & & 3; & 4 & ↊ & 9 \\ \hline
 & & & 2 & 6 & 3 & ↊ & 6 & 9 \\
 & & 2 & 9 & 8 & 3 & 8 & ↊ & 0 \\
 & 1 & 1 & 5 & 8 & 8 & 4 & 0 & 0 \\
+ & ↊ & 1 & 3 & 6 & 3 & 0 & 0 & 0 \\ \hline
 & ↋ & 5 & 9 & 5 & 6 & ↋ & 4 & 9
\end{array}$$

(Notice that the carries for each of the digits of the multiplier are on a separate line; and when we've finished multiplying that digit, the carries are struck out so they will not be confused with the new ones.)

So we pretend that the dits are not there, and solve the problem otherwise as normal. Then, when the problem is complete, *we count the total number of fractional digits in the factors*. So in the first factor, we have two fractional digits ("21"), and the in second factor we have three fractional digits ("4↊9"). Two digits and three digits is five total digits.

Then we "point out" the same number of digits in our answer; that is, five ("56↋49"):

$$\begin{array}{r} 3\;4\;5;2\;1 \\ \times \qquad\quad 3;4\;↊\;9 \\ \hline ↋\;5\;9;5\;6\;↋\;4\;9 \end{array}$$

The number of fractional digits in the answer is equal to the total number of fractional digits in the factors. As said above, *slightly* more complicated than in addition and subtraction; but only slightly.

Exercises 5.8

67. $\begin{array}{r} 4;8 \\ \times \quad 3 \\ \hline \end{array}$ Answer **68.** $\begin{array}{r} 4;8 \\ \times\, 3;3 \\ \hline \end{array}$ Answer **69.** $\begin{array}{r} 4\;3\;2\;8;↊ \\ \times \quad 3;4\;5 \\ \hline \end{array}$ Answer **6↊.** $\begin{array}{r} 3\;↋\;↊;4\;9 \\ \times \quad ↊;4\;7\;8 \\ \hline \end{array}$ Answer

5.3.6 Multiplying Negatives

Multiplying negative numbers is quite simple, even simpler than adding and subtracting them. With addition and subtraction we are forced to manipulate the figures to place the signs side by side, and then resolve the different signs into a final result. But to multiply negative numbers, we need only remember two simple rules:

1. Multiplying factors of the *same* sign yields a *positive* product.
2. Multiplying factors of *different* signs yields a *negative* product.

So a negative times a negative is a positive, and a positive times a positive is a positive; but a negative times a positive is a negative, and a positive times a negative is a negative.

Note that, when you have multiple multiplication operations strung together, you must apply these two rules in sequence. E.g., $-4 \times -4 \times 2 \times -3$ will result in a negative, because -4×-4 will equal a positive, 14; 14×2 will equal a positive, 28; and then 28×-3 will equal a negative, -80.

Besides these two simple rules, the algorithm for doing the problems involving negative numbers is the same. It doesn't get much simpler than that.

Exercises 5.9

6↋. Give the sign (positive or negative) of the products in the following equations. Answer

6↋(a) -4×3. Answer **6↋(b)** 4×3. Answer **6↋(c)** -4×-3. Answer
6↋(d) 4×-3. Answer **6↋(e)** -17×57. Answer **6↋(f)** 4;↋49 × 8;2. Answer
6↋(g) $-33;33 \times -4;5713$. Answer

5.4 Division

Division, as an operation, is done differently from the others, and consequently will take a little time to understand. Unlike addition and multiplication, but like subtraction, it is *anticommutative*, so the order of the numbers in a division problem is important, and changing it will change the answer. It is essentially the inverse of multiplication.

Division uses several symbols; for now, we will use $\div$.

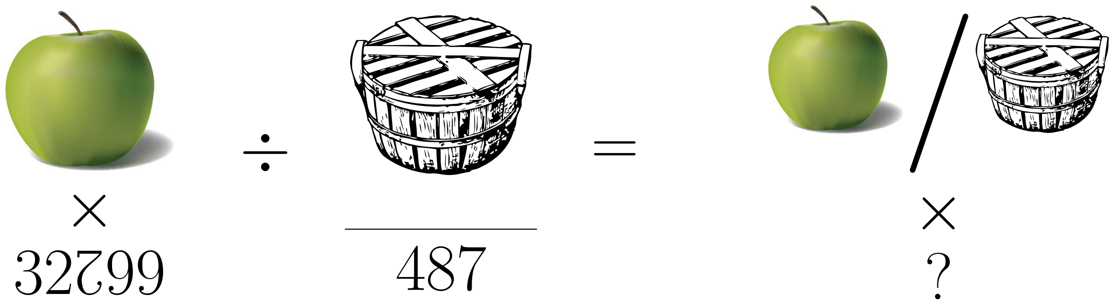
Multiplication is repeated addition, or totalling multiple groups of a certain number; *division* is taking that total and breaking it up into multiple groups. In other words,

$$a \times b = c \qquad c \div b = a \qquad c \div a = b$$

all mean the same thing, the same way that $a + b = c$, $c - b = a$, and $c - a = b$ all mean the same thing.

Let's continue with the apple examples and get a concrete example of what division really is. Say we have an orchard that we know produces (on average) 32↊99 apples every year. We have 487 bushels. When it comes time for the harvest, we wish to know how many apples will go into each bushel, to ensure that we have enough bushels to hold them all.

Notice that this is a multiplication problem, but backwards; rather than finding the total number of apples based on the number of bushels and the number of apples in each, we already know the total number of apples and the number of bushels, and wish to find the number of apples that we'll need to put in each.



Most readers will be familiar with notation like "mi/hr" for "miles per hour" (or "km/hr," as the case may be). We pronounce that "/" as *per*; that's because what we're really doing there is *division*. So the above diagram literally means "*apples divided by bushels equals apples per bushel*." That's what division is.

Division can be written in many ways, just as multiplication can. The most important are these:

$$a \div b \qquad a/b \qquad \frac{a}{b} \qquad b\overline{)a}$$

That last one looks odd, but it's important because it's the form we use for *long division*.[9]

Because division is *anticommutative*, the terms we use for each part of the problem are important. So it's important to learn these terms correctly; you can match them up with the colors above.

dividend

the whole from which the divisor will be taken in a division operation; the first or top number, or the number under the line in a long division problem; red in the above equations.

divisor

the part which will be taken from the whole in a division operation; the second or bottom number, or the number to the left of the line in a long division problem; blue in the above equations.

quotient

the answer in a division operation; that is, the number of times the divisor occurs in the dividend.

In short, the *dividend* divided by the *divisor* gives us the *quotient*.

5.4.1 Two Perspectives on Division

Division is unique among the four functions in that it can be viewed in two ways; these two ways are different, yet both completely correct. Solving problems that involve division might require the budding arithmetician to look at those problems in one or the other of these ways. The two perspectives are of division as *partitive* and as *quotative*.

5.4.1.1 Division as Partitive

The *partitive* view sees division as being about *partitioning*; that is, as trying to find the number of equal groups of the size of the divisor that fit into the dividend. From this perspective, the *dividend* is the total size of the group; the *divisor* is the number of groups we want to form out of that dividend; and the *quotient* is the size of each group that would be formed.

So, for example, when we have a total number of apples and the total number of bushels, and we want to find out how many apples will be in each bushel, we are seeing division as partitive.

[9] *See infra*, Section 5.4.6, at 72.

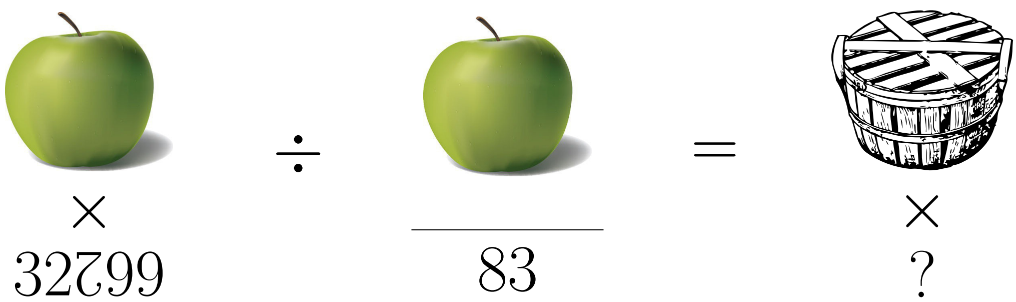
This concept is depicted pictorially above. We know our total number of apples, and we know how many bushels we have; we want to know how many apples will go in each bushel. In arithmetic, we say this as "32799 apples divided by 830 bushels equals the number of apples per bushel."

We're viewing division as partitive when we try to divide a larger whole into a known number of groups. E.g., we know the number of students and the number of teachers, and want to ensure each teacher has the same number of students.

5.4.1.2 Division as Quotative

The *quotative* view sees division as being about *grouping*; that is, as trying to determine how many groups of the size of the divisor can be formed out of the dividend. In other words, the *dividend* is the total number of items, and the *divisor* is the size of each group, and the *quotient* is the number of groups we end up with.

So, for example, when we have a certain number of apples and the number of apples that fit into a bushel, and we want to know how many bushel baskets we need, we are seeing division as quotative.



This concept is depicted pictorially above. We know the total number of apples, and we know how many apples go into each bushel; we simply need to know how many bushels we'll have. In arithmetic, this problem is read, "32799 apples divided by 83 apples per bushel equals the number of bushels."

We're viewing division as quotative when we know the size of the groups we want, and need to know how many of those groups we'll have. The partitive-quotative concepts are illustrated in the table below:

	Partitive	*Quotative*
Problem	$10 \div 3 = 4$	$10 \div 3 = 4$
Conception	10 broken into 3 groups; what is the size of each group?	10 broken up into groups of 3; how many groups of three?

Again, these two perspectives on division are just that: perspectives. Both are perfectly correct statements of what we're doing in division; they differ only in the intent of the problem solver.

Finally, when we have finished dividing the dividend by the divisor, sometimes we have some left over:

> ***remainder***
>
> when the maximum number of groups the size of the divisor have been fit into the dividend (giving the quotient), the remainder is the number left over; by necessity, lesser than the divisor

Remainders will become more important when we discuss *modulation* later on.[7]

Exercises 5.7

70. Write the division problem which describes the following questions. Do *not* worry about solving it. Answer

70(a) How many 2 are contained in 13? Answer

70(b) What part of 4 is 3? Answer

70(c) How many thirds are in 8? Answer

70(d) How many $\frac{2}{3}$ are in $\frac{3}{4}$? Answer

70(e) You have 356 students in a grade and 8 teachers; each teacher should have, as nearly as possible, the same number of students. How many students will each have? Answer

70(f) You have \$18, which must be divided up among 24 friends. How much will each friend receive? Answer

5.4.2 Simple Division

Ultimately, as discussed above, division is just multiplication done backwards. So often, we can conduct division the same way we conduct multiplication: by the use of a multiplication table.[8]

Just as we saw that we can do subtraction by using the addition table in reverse,[10] we can do division by using the multiplication table in reverse. Let's take a look at the same problem:

[7] *See infra*, Section 5.4.4, at 69. [8] *See supra*, Section 5.3.2, at 55. [10] *See supra*, Section 5.2.2, at 46.

$$\begin{array}{r} 7 \\ \times\ 9 \\ \hline 53 \end{array} \qquad \begin{array}{r} 9 \\ \times\ 7 \\ \hline 53 \end{array} \qquad \begin{array}{r} 53 \\ \div\ \ 9 \\ \hline 7 \end{array} \qquad \begin{array}{r} 53 \\ \div\ \ 7 \\ \hline 9 \end{array}$$

We can see that all four of these equations are equivalent by using the table below:

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	1	2	3	4	5	6	7	8	9	↊	↋	10
2	2	4	6	8	↊	10	12	14	16	18	1↊	20
3	3	6	9	10	13	16	19	20	23	26	29	30
4	4	8	10	14	18	20	24	28	30	34	38	40
5	5	↊	13	18	21	26	2↋	34	39	42	47	50
6	6	10	16	20	26	30	36	40	46	50	56	60
7	7	12	19	24	2↋	36	41	48	53	5↊	65	70
8	8	14	20	28	34	40	48	54	60	68	74	80
9	9	16	23	30	39	46	53	60	69	76	83	90
↊	↊	18	26	34	42	50	5↊	68	76	84	92	↊0
↋	↋	1↊	29	38	47	56	65	74	83	92	↊1	↋0
10	10	20	30	40	50	60	70	80	90	↊0	↋0	100

This will suit us for division problems which fall on this table. However, many will not; and for that, we must turn to *long division*.[11] First, though, there are some ancillary concepts we must discuss.

Exercises 5.↋

71. 46 ÷ 6 Answer **72.** 23 ÷ 9 Answer **73.** 84 ÷ ↊ Answer **74.** 34 ÷ 5 Answer
75. 56 ÷ ↋ Answer **76.** 53 ÷ 7 Answer **77.** ↊1 ÷ ↋ Answer **78.** 48 ÷ 8 Answer
79. 92 ÷ ↊ Answer **7↊.** 5↊ ÷ 7 Answer **7↋.** 13 ÷ 5 Answer **80.** 40 ÷ 6 Answer

5.4.3 Division Facts

Just as with multiplication, there are a few facts about division that it would be well to remember. These facts are critical to understanding how division functions, and will be an immense help in shortcutting some of the more detailed calculations. It will further help to make short division[12] even shorter.

The first is that division's identity element is, insofar as it has one, 1; however, like that of subtraction, this is a *right-identity* only. That is, any number divided by 1 equals the number itself; but 1 divided by any number does *not* equal the number itself. Indeed, 1 divided by a number is referred to as the *reciprocal* of that number:

[11] *See infra*, Section 5.4.6, at 72. [12] *See infra*, Section 5.4.7, at 84.

reciprocal

a number which, when multiplied by another, yields a product of 1; the number which equals one divided by another number

These reciprocals are important because, when you multiply a number by its reciprocal, the product is 1.

The next important thing to remember is that *it is impossible to divide by zero*. It's not just difficult; it literally doesn't work. This fact makes sense when we consider what division is: how can we separate a whole into groups of zero each? Or, alternatively, how can we separate a whole into zero groups of a certain size? The concept just doesn't make sense.

So we say that dividing by zero is *undefined*; that is, the function of division simply doesn't cover that situation.

Next, any number divided by itself equals 1. This, again, makes sense when we consider the nature of division: you can divide a group up into only one group of the same size. So all numbers divided by themselves equal 1.

Next, we must mention the ease of dividing by a 1 followed by any number of zeroes. We've already seen in multiplication that this is a simple matter of moving the dit to the right[13]; in division, it's a simple matter of moving the dit to the *left*; as before, we move it an equal number of places as there are zeroes in the divisor:

$$6\text{↊}54;8\text{↋}\text{↊}7 \div 1000 = 6;\text{↊}548\text{↋}\text{↊}7$$

Also, as before, if we need to move it more places than there are digits, we just add zeroes, plus an extra one to go on the left of the dit:

$$6\text{↊}54;8\text{↋}\text{↊}7 \div 1000000 = 0;006\text{↊}548\text{↋}\text{↊}7$$

We can also do the same tricks with other numbers followed by a string of zeroes; for example, when dividing by 200, we can divide the problem into two parts: first dividing by 100, then dividing that quotient by 2, to take one difficult problem and turn it into two easy ones.

Finally, division by negative numbers follows the same rules as multiplication by negative numbers: if the signs are the same, the result is positive; but if the signs are different, the result is negative. Otherwise, though, the actual algorithms work the same way.

So, to sum up:

- Division by 1 equals the dividend; but not the other way around.
- Division by 0 doesn't work, and is undefined.
- When the dividend and divisor are the same number, the quotient is 1.

[13] *See supra*, Section 5.3.3, at 56.

- Division by a 1 followed by one or more zeroes is simply moving the dit to the left a number of places equal to the number of zeroes in the divisor.

Now a few exercises to test our comprehension.

Exercises 5.10

81. $348 \div 0$ Answer **82.** $873 \div 100$ Answer **83.** $4;839 \div 10000$ Answer **84.** $4;8 \div 1$ Answer **85.** 0;00↊ ÷ 1000 Answer **86.** ↋;643 ÷ 10 Answer **87.** $578 \div 578$ Answer

5.4.4 Modulation

It will quickly be seen, however, that with division, not all of our answers come out round. In the other three functions (addition, subtraction, and multiplication), every problem comes up with a definite, *integral* answer (at least, if we start with two integers). With division, though, that isn't always the case. Take a look at the multiplication table: what is $19 \div 2$, or ↊4 ÷ ↋?

When we view division as *partitive*, we see the divisor as the number of groups we want to form out of the dividend, and the quotient is the size of these groups. Sometimes (quite often, in fact), the dividend can't be broken into a number of groups equal to the divisor of *any* size (that is, with any quotient) without some being *left over*.

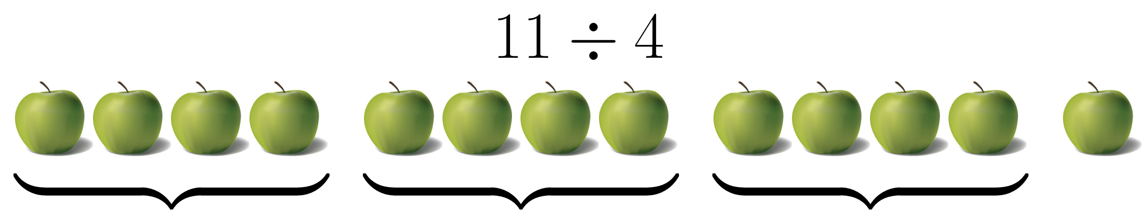
Similarly, when we view division as *quotative*, we see the divisor as the size of the groups we want to make, and the quotient is the number of those groups. Sometimes (quite often, in fact), in this case, too, there will be some *left over*.

In any case, when a division problem does not yield an even answer (when there are some left over after we're done), we are faced with a choice: produce a fraction, or calculate the *remainder*. The latter, calculating the remainder, is referred to as *modulation*, and is often pronounced "mod"; e.g., "3↋ mod 5 equals 2."

Let's consider the matter with our now-trusty batch of apples. We have 11 apples:

And we have a number of small packages, each of which can hold 4 apples. We want to put this batch of 11 apples into these 4-apple packages.

What we're doing here is dividing 11 by 4; in mathematical terms, this is $11 \div 4$. So let's divide our 11 apples into groups of 4:



We have 3 groups of 4; therefore, $11 \div 4 = 3$. However, we also have one apple left over. This leftover apple is called, appropriately, the *remainder*.

When the remainder is what we really *want* from the calculation, rather than the quotient, we do *modulation*:

> ***modulation***
>
> the arithmetical operation in which the remainder of a division problem is the result; symbol is "%" or "mod"

As mentioned above, this operation is typically pronounced "mod," and can be represented symbolically by either the word "mod" or by a perbiqua sign, "%":

$$11 \;\%\; 4 = 1 \qquad\qquad 11 \mod 4 = 1$$

The algorithm for modulation is quite simple:

- Find the quotient which comes as close as possible to the dividend without going over it. E.g., if your problem is 2↋ % 5, the closest quotient you'll get is 7, because 5 × 7 = 2↋, while 5 × 8 is higher than the dividend, 2↋. If your problem is 30 % 5, you have the same quotient, because 5 × 7 = 2↋, but 5 × 8 = 34, which is higher than the dividend, 30.

- Take the difference of your dividend and the product of your quotient and your divisor. E.g., if your problem is 2↋ % 5, your quotient is 7; 7 × 5 = 2↋, and 2↋ − 2↋ = 0, so your remainder is 0. But if your problem is 30 % 5, your quotient is still 7; 7 × 5 = 2↋, and 30 − 2↋ = 1, so your remainder is 1.

We can also, of course, state the answer to the division problem (not the modulation problem) as a whole number plus some fractional part, with no remainder. This is an option that comes about through long division, which is the subject of Section 5.4.6 on page 72. We will discuss it at the appropriate time. Stating only the integral part of the quotient, with or without the remainder, is called *integer division*:

> ***integer division***
>
> division in which only the integral part of the quotient, or the integral part with a remainder, are stated, ignoring any fractional part

Most of our work in this text will not be integer division, but will involve determining the fractional part of the quotient, if any. But integer division is a concept that comes up from time to time, and so should be learned, at least in connection to modulation.

Exercises 5.11

88. Perform the requested modulation. Use visual aids, finger-counting, the multiplication table, or whatever other trick will help you. Answer

88(a) 2↊ % 5 Answer **88(b)** 85 % ↊ Answer **88(c)** 56 % 7 Answer
88(d) 84 % 9 Answer **88(e)** 42 % 6 Answer **88(f)** 84 % ↊ Answer

5.4.5 Factors

We've already seen that *factors* are the numbers we use to find a product in a multiplication problem. However, since division is just reversed multiplication, we find ourselves using the term "factor" quite a bit when we're dividing. Specifically, we use it to mean a number which, when multiplied by another, will equal our dividend.

> ***factor***
>
> a divisor which exactly divides a dividend, without any remainder

Due to multiplication and division being so closely related, these two uses of the word *factor* are substantially identical; they're just focusing on different aspects of the same thing.

So when we say that we're *factoring* a number, we mean that we are finding its *factors*; that is, we're determining what numbers will exactly divide it without any remainder. Any given number may have very few factors, or it may have many. Facility with identifying factors is not *necessary* for long division, but it is certainly very helpful; and fortunately, counting in twelves gives us a lot of help in doing so.

When a certain number x has another number y as a factor, we say that "x has y as a factor," or "x is divisible by y," more commonly the latter.

So let's see a few of the rules of finding factors:

- Any number which ends in 0, 2, 4, 6, 8, or ↊ is called *even*, and is divisible by 2. Numbers which are not even, and therefore not divisible by 2, are called *odd*.
- Any number which ends in 0, 3, 6, or 9 is divisible by 3.
- Any number which ends in 0, 4, or 8 is divisible by 4.
- Any number which ends in 0 or 6 is divisible by 6.
- Any number which ends in 0 is divisible by 10.
- Any number in which:
  - The second-to-last digit is *even* and the last is 0 or 8; or
  - The second-to-last digit is *odd* and the last is 4

  is divisible by 8.
- Any number in which:
  - The second-to-last digit is divisible by 3, and the last digit is 0 or 9; or
  - The second-to-last digit is one which is divisible by 3 plus 1 (e.g., 1, 4, 7, or ↊), and the last digit is 6; or

Divisor Tests	
Divisible by	If
2	Last is 0, 2, 4, 6, 8, or ↊
3	Last is 0, 3, 6, 9
4	Last is 0, 4, 8
6	Last is 0, 6
8	Second-to-last digit divides by 2 and last is 0 or 8 Second-to-last digit doesn't divide by 2 and last is 4
9	Second-to-last digit is 0, 3, 6, or 9 and last is 0 or 9 Second-to-last digit is 1, 4, 7, or ↊ and last is 6 Second-to-last digit is 2, 5, 8, or ↋ and last is 3.
↋	Sum of digits is divisible by ↋; sum the digits of the sum, as many times as needed
10	Last is 0

Table 5.1: Simple Divisibility Tests

- The second-to-last digit is one which is divisible by 3 plus 2 (e.g., 2, 5, 8, or ↋), and the last digit is 3

is divisible by 9.

- If the sum of the digits is divisible by ↋, then the number is itself divisible by ↋. If you can't tell if the sum of the digits is divisible by ↋, you can continue summing the digits of the sum as many times as necessary.

There are divisibility tests for 5, 7, and ↊, as well, though these are more complicated and best left until a greater facility with mental arithmetic has been obtained. The simple rules above, however, will serve well for the time being; they are summarized in Table 5.1 on page 70. These rules are best committed to memory; fortunately, they are simple, and they cover a large majority of cases.

There are a number of special types of numbers, too, which are defined by the presence or absence of certain factors. We've already seen *even* and *odd* numbers:

even

a number divisible by two; its inverse, a number not divisible by two, is said to be *odd*

The divisibility of a number by 2 is also called its *parity*; so an even number has a parity of 0, and an odd number has a parity of 1.

Some numbers are special because of their *lack of divisors*; these are called *prime* numbers:

prime

A natural number greater than 1 which is divisible by only itself and 1

Primes are very important in certain applications, particularly cryptography; for our purposes, they are mostly useful for knowing when a number cannot be further factored. We will see how important this is in finding *greatest common factors*[14] and in reducing fractions.[15]

It follows from this that all primes are odd; all even numbers, of course, divide not only by 1, but by 2. The sole exception to this is 2 itself, which is the only even prime among an infinity of numbers and of primes.

The opposite of prime is *composite*:

composite

a natural number greater than 1 which is not prime; that is, which has factors besides 1 and itself

All composites can be *factored* into a series of primes; that is, each composite number can be divided into its prime factors. You may have to list a prime factor more than once; for example, 20's prime factors are 2, 2, 2, 2, and 3 ($2 \times 2 \times 2 \times 2 \times 3 = 20$). This doesn't need to be done if you're listing simple factors; e.g., 20's factors are 2, 3, 4, 6, 8, and 10. This sort of factoring is an important part of arithmetic, and we'll learn more about it later.[16]

Finally, some numbers are not merely composite, but have so many factors that this becomes one of their chief characteristics. Some of these numbers are called *abundant*, and others are called *highly composite*.

abundant

a number for which the sum of its factors, excluding itself, is greater than the number itself

The lowest abundant number is 10, which has as factors 1, 2, 3, 4, and 6. Note that the sum of these ($1 + 2 + 3 + 4 + 6$) is 14, higher than 10.

highly composite

a positive integer with more factors than any smaller integer

An example here is, again, 10; with with 1, 2, 3, 4, 6, and 10, 10 has more factors than any number lesser than 10; indeed, one must go all the way to 20 to find a number with more factors.

[14] *See supra*, Section 5.5.1.1, at 87. [15] *See supra*, Section 5.5.1.5, at page 94. [16] *See supra*, Section 5.4.9, at 86.

Now that we know about factors, let's move on and talk about *factoring*: that is, finding the factors of a number. But first, a little practice.

Exercises 5.12

89. List the factors of the following numbers. Answer
89(a) 10 Answer **89(b)** 8 Answer **89(c)** 12 Answer **89(d)** 14 Answer
89(e) 11 Answer **89(f)** 36 Answer **89(g)** 4 Answer

8↊. List the *prime* factors of the following numbers. Answer
8↊(a) 10 Answer **8↊(b)** 8 Answer **8↊(c)** 12 Answer **8↊(d)** 14 Answer
8↊(e) 11 Answer **8↊(f)** 36 Answer **8↊(g)** 4 Answer

8↋. You are having a party and are trying to decide how many people you can invite while still having enough food. Specifically, of course, enough *sweets*. Assume you won't be inviting more than 10 people (it's a small party). Answer

8↋(a) If you have 46 cupcakes, how many different numbers of people can you invite if everyone is to get an equal number of cupcakes? Answer

8↋(b) If you have 84 cupcakes, how many different numbers of people can you invite if everyone is to get an equal number of cupcakes? Answer

8↋(c) If you have 34 cupcakes, how many different numbers of people can you invite if everyone is to get an equal number of cupcakes? Answer

8↋(d) If you have 60 cupcakes, how many different numbers of people can you invite if everyone is to get an equal number of cupcakes? Answer

5.4.6 Long Division

Long division is so called because it is just that: long and division. It is legendary among arithmetic students, particularly younger ones, as being difficult and tedious. But truly, with a little practice, it is not much work; and knowing it, along with the rest of the arithmetic in this chapter, will give you the tools you need to progress through the rest of arithmetic and mathematics, so learning long division is well worth the effort.

First, a word about the format. Up until now, we have done our arithmetic with the *operands* placed one on top of the other, displaying the *operation* next to the last one, with the final result on the bottom; e.g.:

$$\begin{array}{r} 3 \\ +\,4 \\ \hline 7 \end{array}$$

With long division, however, we instead arrange the operands in a *tableau*, with the *dividend* underneath a special symbol, the *divisor* to the left of it, and the *quotient* on the *top*; e.g.:

$$\begin{array}{r} 9 \\ 5\,\overline{)\,3\;9} \end{array}$$

We do our work *beneath* this tableau, rather than above it; and that is the reason for this unusual format: it leaves us with plenty of space to do our work, which at least sometimes we will need. So let us now proceed and see what to do.

We'll start with a relatively simple problem:

$$4\overline{)\,3\;9\;4\;8}$$

This problem is pronounced "3948 divided by 4." 3948 is the *dividend* and 4 is the *divisor*.

Now, just as with the other four functions, our algorithm requires us to attack this problem one digit at a time. In long division, though, we begin with the first digit on the left, not the first digit on the *right*; so here we begin with the 3.

$$4\overline{)\,3\;9\;4\;8}$$

When we isolate part of our dividend in this way, it is called the *partial dividend*. So here, our partial dividend is 3.

Now that we've isolated our partial dividend, we take our *divisor* (the number to the left) and determine how many times the divisor goes into that partial dividend; in this case, how many times 4 goes into 3. That's just another way of asking, "Is 3 larger than 4; and if so, how many 4s go into it?" We can stop there; 3 is not larger than 4, and consequently it doesn't *go into* 4; that is, we cannot divide 3 by 4 and get a quotient greater than 1. So we need to move on.

$$4\overline{)\,3\;9\;4\;8}$$

Now we look at the first *two* digits of the dividend, making that our partial dividend; in this case, 39. So we ask the same question: how many times does 4 go into 39? Or, is 39 larger than 4; and if so, how many 4s go into it? To answer that question, we can do simple division and integer division as explained above.[17] Look at the multiplication table, and find the column for the divisor, 4; then find the number which is closest to 39 without going over it, in this case 38; then find what 38 ÷ 4 is, in this case Ɛ.

That Ɛ gives us the leftmost digit of our quotient. So we place it *above the last digit of the partial dividend*; in this case, above the 9 in 39:

$$\begin{array}{r} \text{Ɛ}\;\;\;\;\; \\ 4\overline{)\,3\;9\;4\;8} \end{array}$$

Now we get to utilize the space *underneath* our problem. We take the Ɛ and multiply it by 4, giving us 38, as we saw in our multiplication table in order to get our digit 9 in the first place. We then place that 38 underneath the 39 in our problem:

[17] *See supra*, Section 5.4.2, at 66; and at 6↊.

$$\begin{array}{rllll} & & \text{Ɛ} & & \\ \cline{2-5} 4) & 3 & 9 & 4 & 8 \\ & 3 & 8 & & \end{array}$$

Remember that we arrive at the number by calculating Ɛ × 4 = 38; so we're not doing anything new, just combining old things in a different way.

Now we find the *difference* between the partial dividend (39) and the number we calculated (38), like so:

$$\begin{array}{rllll} & & \text{Ɛ} & & \\ \cline{2-5} 4) & 3 & 9 & 4 & 8 \\ & 3 & 8 & & \\ & & 1 & & \end{array}$$

Note that this is simply finding the remainder of 39 ÷ 4; that is, this is simply finding 39 % 4.

Now we need to work with the next digit in our dividend, the 4 (not the divisor; the 4 which lies between the 9 and the 8). But we don't have any room to do that; and besides, we've already taken care of the 39, and reduced them to 1 down below. Therefore we *drop* the 4 to be next to the 1 which is left over from the 39; this gives us a 14:

$$\begin{array}{rllll} & & \text{Ɛ} & & \\ \cline{2-5} 4) & 3 & 9 & 4 & 8 \\ & 3 & 8 & & \\ & & 1 & 4 & \end{array}$$

This 14 is now our partial dividend. So we treat the 14 the same way we treated the 39 originally: we ask how many times 4 goes into it. Or, alternatively, what is 14 ÷ 4? In this case, we get an exact answer: 14 ÷ 4 = 4. So 4 is the next digit of our quotient, and we fill it in:

$$\begin{array}{rllll} & & \text{Ɛ} & 4 & \\ \cline{2-5} 4) & 3 & 9 & 4 & 8 \\ & 3 & 8 & & \\ & & 1 & 4 & \end{array}$$

Then we calculate that digit multiplied by the divisor; in this case, 4 × 4, which is 14, and we put it down on the bottom:

$$
\begin{array}{r|cccc}
 & & \text{Ɛ} & 4 & \\ \hline
4 & 3 & 9 & 4 & 8 \\
 & 3 & 8 & & \\
 & & 1 & 4 & \\
 & & 1 & 4 &
\end{array}
$$

Then we find the difference between the two:

$$
\begin{array}{r|cccc}
 & & \text{Ɛ} & 4 & \\ \hline
4 & 3 & 9 & 4 & 8 \\
 & 3 & 8 & & \\
 & & 1 & 4 & \\
 & & 1 & 4 & \\
 & & & 0 &
\end{array}
$$

Now we need to get the next digit of the dividend into the problem; that next digit is also the last, 8. So we *drop* the 8 down next to the 0:

$$
\begin{array}{r|cccc}
 & & \text{Ɛ} & 4 & \\ \hline
4 & 3 & 9 & 4 & 8 \\
 & 3 & 8 & & \\
 & & 1 & 4 & \\
 & & 1 & 4 & \\
 & & & 0 & 8
\end{array}
$$

Now that 8 is our partial dividend, and we see how many times 4 goes into 8; or we calculate $8 \div 4$, which is 2. So we put that into our answer as the next digit of our quotient:

$$
\begin{array}{r|cccc}
 & & \text{Ɛ} & 4 & 2 \\ \hline
4 & 3 & 9 & 4 & 8 \\
 & 3 & 8 & & \\
 & & 1 & 4 & \\
 & & 1 & 4 & \\
 & & & 0 & 8
\end{array}
$$

Now, we multiply 2 by our divisor, 4, and get 8; we put this at the bottom, and then find the difference:

$$
\begin{array}{r}
↋\,4\,2 \\
4\,\overline{)\,3\,9\,4\,8} \\
3\,8 \\
1\,4 \\
1\,4 \\
0\,8 \\
8 \\
0
\end{array}
$$

That's 0, and we're at the end of our problem now; that is, our partial dividend is 0, and we have no more digits to drop other than 0 (the imaginary zeroes after the dit of the dividend, in this case, which we can add without changing the value of the number). So we're done. $3948 \div 4 = ↋42$.

To sum up, here are the steps we had to go through to solve this problem:

1. 4 does not go into 3, so we move to the next digit.
2. 4 goes into 39 ↋ times, with one left over. So put the ↋ above the 39.
3. $↋ \times 4 = 38$, so put the 38 below the 39.
4. $39 - 38 = 1$.
5. Drop the 4, giving us 14.
6. 4 goes into 14 4 times, with none left over. Put the 4 above the 4 in the dividend.
7. $4 \times 4 = 14$, so put the 14 below the other 14.
8. $14 - 14 = 0$.
9. Drop the 8, giving us 08, or simply 8.

↊. 4 goes into 8 twice; so put the 2 above the 8 in the dividend.

↋. $2 \times 4 = 8$, s put the 8 below the other 8.

10. $8 - 8 = 0$. We're done.

Hence the system's name of *long* division. But still, none of these steps is particularly challenging; there are just a lot of them. The important thing to remember is to isolate the partial dividend and always work with that, never with the full dividend that you start with. In this way, by accumulating the results of simple problems, you will find the solution to the difficult one you started with.

Long division is a skill, like most arithmetic, which is best learned by practice. So let's take a look at a few more examples.

Examples

To begin, we will try two problems which do not have any remainder; that is, problems with dividends which are divided evenly by their divisors. Long division can handle problems much less neat than these, but they are the best to start with.

	2	Ɛ	9	Ɛ
3)	8	Ɛ	5	9
	6			
	2	Ɛ		
	2	9		
		2	5	
		2	3	
			2	9
			2	9
				0

3 goes into 8 twice. Place the 2 above the dividend's 8. $2 \times 3 = 6$*; place the 6 beneath the dividend's 8.* $8 - 6 = 2$*. Drop the dividend's Ɛ to sit beside the 2, making the number 2Ɛ. 3 goes into 2Ɛ Ɛ times. Place that Ɛ above the Ɛ in the dividend.* Ɛ × 3 = 29*; place the 29 beneath the 2Ɛ.* 2Ɛ − 29 = 2*. Drop the 5 to beside that 2, making 25. 3 goes into 25 9 times. Place the 9 above the dividend's 5.* $9 \times 3 = 23$*; place that 23 beneath the 25.* $25 - 23 = 2$*. Drop the dividend's final digit, 9, to beside the 2, making 29. 3 goes into 29 Ɛ times. Place that Ɛ above the dividend's 9.* 9 × 3 = 2Ɛ*; place that 2Ɛ beneath the 2Ɛ.* 2Ɛ − 2Ɛ = 0*. Since there are no more digits in the dividend to drop, we are finished;* 8Ɛ59 ÷ 3 = 2Ɛ9Ɛ.

Next, we will try to do the same thing with a two-digit divisor. Multiple-digit divisors aren't really any different from single-digit divisors, but they *look* different, and frequently they require some intelligent guessing and trial and error.

```
         2 2 2
2 2 ) 4 8 8 4
      4 4
        4 8
        4 4
          4 4
          4 4
            0
```

A two-digit divisor, so a bit trickier to start; some trial and error might be needed to get the first digit of our quotient. 22 goes into 48 twice; put the 2 above the 48 in the quotient. $2 \times 22 = 44$. *Write the 46 beneath the 48.* $48 - 44 = 4$. *Drop the 8 (make sure we drop the* second *8; we've already used the first, as part of 48). So now we've got 48. We already know that 22 goes into 48 twice; so put another 2 in the quotient.* $2 \times 22 = 44$. *Write the 44 beneath the 48, then find the difference.* $48 - 44 = 4$. *Drop the 4, making 44. 22 goes into 44 exactly twice; so place the 2 in the quotient.* $2 \times 22 = 44$*; place the 44 beneath the other 44.* $44 - 44 = 0$. *We have a 0 difference and we've already used the dividend's last digit, so we're finished.* $4884 \div 22 = 222$.

Now let's try another one with a one-digit divisor, just to make sure we've got the process down.

```
    1 0 7 3
7 ) 7 4 2 9
    7
    0 4
      0
      4 2
      4 1
        1 9
        1 9
          0
```

7 goes into 7 once. Put the 1 above the 7. $1 \times 7 = 7$*, so put the 7 beneath the 7 in the dividend.* $7 - 7 = 0$*; write the difference underneath. Drop the 4. 7 goes into 4 zero times ("04" is the same as "4"), so put the zero above the 4 in the dividend.* $0 \times 7 = 0$*, so put the 0 underneath the 4.* $4 - 0 = 4$*; write the difference underneath; drop the 2. 7 goes into 42 7 times, so put the 7 above the 2 in the dividend.* $7 \times 7 = 41$*; put that under the 42.* $42 - 41 = 1$*; write the difference underneath. Drop the 9, making 19. 7 goes into 19 3 times; put the 3 above the 9 in the dividend.* $3 \times 7 = 19$*; write that underneath.* $19 - 19 = 0$*, so we have no remainder.* $7429 \div 7 = 1073$.

Now we try, for the first time, to divide a dividend by a divisor which does not work evenly; that is, we will have a remainder.

```
    1 6 2 2    R 2
7 ) ↊ 7 3 4
    7
    3 7
    3 6
      1 3
      1 2
        1 4
        1 2
          2
```

Seven goes into ↊ once. Put the 1 above the ↊ in the dividend. $1 \times 7 = 7$*, so put the 7 beneath the ↊.* ↊ $- 7 = 3$. *Drop the 7. 7 goes into 37 6 times. Put the 6 above the 7 in the dividend.* $6 \times 7 = 36$. *Put the 36 beneath the 37.* $37 - 36 = 1$. *Drop the 3. 7 goes into 13 twice. Put the 2 above the 3 in the dividend.* $2 \times 7 = 12$*; put the 12 beneath the 13.* $13 - 12 = 1$. *Drop the 4. 7 goes into 14 twice; put the 2 above the 4 in the dividend.* $2 \times 7 = 12$. *Put the 12 underneath the 14.* $14 - 12 = 2$. *We are now at the last digit, so the remaining difference, 2, is the* remainder. ↊734 $\div 7 = 1622$R2*;* ↊734 % $7 = 2$.

As you can see, when we reach the last digit of our dividend, we simply take the remaining partial dividend, 2, and make it our remainder. That remainder is displayed after the quotient with a capital "R" preceding it. Remember that, if what you're really interested in is the remainder and not the quotient, you do the same thing, but the operation is called *modulation*.[18]

Sometimes it is easier to get the even quotient and not worry about the remainder, as we have done here. However, at other times it's important to get a more exact answer by continuing on with the problem, calculating the fractional part for the quotient; our next example will do so with the above problem.

```
     1 6 2 2;3 5 1 8 6 ↊
   ---------------------
7 ) ↊ 7 3 4;0 0 0 0 0 0
    7
    3 7
    3 6
      1 3
      1 2
        1 4
        1 2
          2 0
          1 9
            3 0
            2 ↋
              1 0
                7
                5 0
                4 8
                  4 0
                  3 6
                    6 0
                    5 ↊
                      2
```

Seven goes into ↊ once. Put the 1 above the ↊ in the dividend. 1 × 7 = 7, *so put the 7 beneath the ↊.* ↊ − 7 = 3. *Drop the 7. 7 goes into 37 6 times. Put the 6 above the 7 in the dividend.* 6 × 7 = 36. *Put the 36 beneath the 37.* 37 − 36 = 1. *Drop the 3. 7 goes into 13 twice. Put the 2 above the 3 in the dividend.* 2 × 7 = 12; *put the 12 beneath the 13.* 13 − 12 = 1. *Drop the 4. 7 goes into 14 twice; put the 2 above the 4 in the dividend.* 2 × 7 = 12. *Put the 12 underneath the 14.* 14 − 12 = 2. *We are at the last digit; so now we must drop a* zero *if we are to proceed, and we fill in some zeroes in the dividend to make things clearer. Drop the 0 to sit beside the 2, making 20. 7 goes into 20 3 times. Add the dit to the answer and put the 3 into the quotient.* 3 × 7 = 19; *put the 19 beneath the 20.* 20 − 19 = 3. *Drop another 0, making 30. 7 goes into 30 5 times; add the 5 to the quotient.* 5 × 7 = 2↋. *Put the 2↋ underneath the 30.* 30 − 2↋ = 1. *Drop a 0, making 10. 7 goes into 10 one, so place the 1 in the quotient.* 1 × 7 = 7, *so place the 7 under the 10.* 10 − 7 = 5. *Drop the 0, making 50. 7 goes into 50 8 times. Put the 8 in the quotient.* 8 × 7 = 48; *put the 48 beneath the 50.* 50 − 48 = 4. *Drop the 0, making 40. 7 goes into 40 6 times. Put the 6 in the quotient.* 6 × 7 = 36; *put the 36 beneath the 40.* 40 − 36 = 6. *Drop the 0, making 60. 7 goes into 60 ↊ times; put the ↊ in the quotient.* ↊ × 7 = 5↊; *put the 5↊ beneath the 60.* 60 − 5↊ = 2. ↊734 ÷ 7 = 1622;35186↊ . . .

Notice the ". . . " at the end of our quotient in the explanation on the right side of the page. We still had a 2 left over when we stopped calculating; that means we could have kept going. A long division problem isn't *finished*, strictly speaking, until we get a 0 at the bottom, as we did in our first example (8↋59 ÷ 3); that is, until we get a 0 as our partial dividend, with no non-zero digits of the full dividend left to drop. But we have to stop at some point, and this point seemed as good as any.

This illustrates an important fact about division: unlike the other four functions, not all division problems have exact answers. That is, some will yield an answer which

[18] *See supra*, 5.4.4, at 69.

is a *nonterminating fraction* or a *repeating fraction*. In the above example, we have a repeating fraction; if we continued gathering more digits (which we can do forever, in the same way as we acquired the ones we did), we would see the sequence 35186↊ repeating for as long as we cared to go. That is, we'd see 1622;35186↊ 35186↊ 35186↊ . . ..

In our examples here, we have seen long division done in one case on a multi-digit divisor, and in one case involving fractional parts. Both of these situations, however, require a bit more practice; so we will address each of them in turn. But first, some practice with the simpler case.

Exercises 5.13

90. Divide; give quotient and remainder. Answer

90(a) 4) 4 3 8 Answer **90(b)** 3) ↊ 3 8 Answer **90(c)** ↋) 4 3 ↊ 9 7 Answer

90(d) 7) 9 ↋ 2 2 Answer **90(e)** 5) 2 4 ↊ 5 Answer

90(f) 9) ↋ ↊ 9 8 7 6 Answer

91. You are out to dinner with some friends. Assuming that you have agreed to split the bill evenly, determine how much each of you must pay. The total check is $185;60. Ignore fractional parts and remainders in your answer; you're merely getting an estimate, so you can argue about how much more you're paying than your food actually cost. Answer

91(a) If you are with two other friends. Answer
91(b) If you are with four other friends. Answer
91(c) If you are with eight other friends. Answer

92. You are a manager at a warehouse. Answer

92(a) You have received an order for 355 crates of your product to be sent to 7 different locations. How many crates do you send to each location? Answer
92(b) You realize that you need to order more of a certain product to keep in stock. This product comes in plattes of 12 crates each. To keep up an appropriate level of stock, you need 86 crates. How many plattes do you need to order? Answer
92(c) If you order 7 plattes, how many creates short of your target will you be? Answer
92(d) If you order 8 plattes, how many extra crates will you have? Answer

93. You are grocery-shopping, and you notice that 3 pounds of apples costs $3;60. Answer

93(a) How much does each pound cost? Answer
93(b) How much would seven pounds cost? Answer

5.4.6.1 Multi-Digit Divisors

The problems of multi-digit divisors are really the same as those of single-digit divisors; however, because of their increased complexity, some direct treatment of the problem can be helpful.

Single-digit divisors are comparatively easy because we have already memorized our multiplication table[19]; this means that finding the right candidates for each digit of the quotient is a simple matter of mentally looking up the divisor's column in the table and finding the closest dividend that does not exceed it, as we did in our section on simple division above.[1↊] But it's neither possible nor practical to memorize all two-digit, three-digit, and so forth tables; so we have to resort to other means.

Largely, this "other means" is that dullest of mathematical tools: trial and error. Let's take an example:

$$8\,3\overline{)\,7\,4\,↊\,3\,9}$$

We do this problem in the same way as we do any other long division problem; but the arithmetic is more difficult. Here, we take our divisor, 83, and proceed through the digits of the dividend starting on the left, seeing if we can fit at least one 83 into them; in other words, until we have a number greater than or equal to our divisor. We are, in other words, finding our partial dividend.

83 is *larger* than 7, so we need at least two digits; 83 is *larger* than 74, so we need at least three digits; 83 is *smaller* than 74↊, so we can pause here. How many times does 83 go into 74↊? In other words, what is 74↊ ÷ 83?

Well, we have to do trial and error here. First, we must remember that 83 must go into 74↊ at least 1 time and at most ↋ times. Then, we can *estimate* to get in the right ballpark before we start testing numbers out. If our divisor were 100, we know that the answer would be 7; since 83 is quite a bit smaller than 100, we know the true answer must be more than 7, so let's start at 8. Then, we just try it out:

$$\begin{array}{r} {}^{2}\\ 8\,3\\ \times\quad 8\\ \hline 5\,6\,0 \end{array} \qquad \begin{array}{r} {}^{2}\\ 8\,3\\ \times\quad 9\\ \hline 6\,2\,3 \end{array} \qquad \begin{array}{r} {}^{2}\\ 8\,3\\ \times\quad ↊\\ \hline 6\,↊\,6 \end{array} \qquad \begin{array}{r} {}^{2}\\ 8\,3\\ \times\quad ↋\\ \hline 7\,6\,9 \end{array}$$

We see that 83 × ↊ = 6↊6, which is *lesser than* this portion of our dividend; but that 83 × ↋ = 769, which is *greater than* this portion of our dividend. So ↊ is our number, which we place in the quotient above the final digit of our partial dividend:

$$\begin{array}{r} ↊\\ 8\,3\overline{)\,7\,4\,↊\,3\,9} \end{array}$$

And, of course, we also know that ↊ × 83 = 6↊6, so we put that beneath our 74↊:

[19] *See supra*, Section 5.3.2, at 55. [1↊] *See supra*, Section 5.4.2, at 66.

$$
\begin{array}{rr}
 & \text{↊} \\
83 & \overline{)\,74\text{↊}39} \\
 & 6\text{↊}6
\end{array}
$$

We find the difference, drop the 3, then begin the whole process over again with a new partial dividend.

$$
\begin{array}{rr}
 & \text{↊} \\
83 & \overline{)\,74\text{↊}39} \\
 & 6\text{↊}6 \\
 & 643
\end{array}
$$

Dropping the 3 gives us 643, our new partial dividend. How many times does 83 go into 643? Or, what is $643 \div 83$?

Again, trial and error. We start by *estimating* the answer. If our divisor were 100, our answer would be 6; since 83 is quite a bit smaller than 100, we'll start with something higher than that; say, 8. Fortunately for us, this range is quite similar to what we did for that last digit; and in fact, we already know the answer. $83 \times 9 = 623$, which is *less* than 643; and $83 \times \text{↊} = 6\text{↊}6$, which is *more* than 643. So the next digit of our quotient is 9:

$$
\begin{array}{rr}
 & \text{↊}9 \\
83 & \overline{)\,74\text{↊}39} \\
 & 6\text{↊}6 \\
 & 643
\end{array}
$$

$9 \times 83 = 623$, so we put that beneath 643, find the difference, then drop the next digit in the dividend, 9.

```
          Ƨ 9
83 ) 7 4 Ƨ 3 9
     6 Ƨ 6
       6 4 3
       6 2 3
         2 0 9
```

Moving on, we see that we must now determine how many times 83 goes into 209. 209 is quite a small number compared to our other dividends. We know that $8 \times 3 = 20$, so we know that $80 \times 3 = 200$. That seems quite close, so let's put in a 3 as our last digit and see if it works for us: is 83×3 less than or equal to 209?

$$\begin{array}{r} 83 \\ \times\quad 3 \\ \hline 209 \end{array}$$

It's exact! That's a stroke of luck; we don't need to worry about remainders or fractionals. So we put it in the quotient:

```
          Ƨ 9 3
83 ) 7 4 Ƨ 3 9
     6 Ƨ 6
       6 4 3
       6 2 3
         2 0 9
         2 0 9
             0
```

As we can see, the process of long division with multiple-digit divisors can be labor-intensive. But the cultivation of good estimation skills can significantly reduce the necessary work, and either way, it is a necessary part of arithmetic.

Exercises 5.14

94. Divide, giving quotient and remainder. Answer

94(a) 14)438 Answer **94(b)** ↊4)2787 Answer **94(c)** 48)9438 Answer

94(d) 697)↊↋987 Answer **94(e)** 5↋49)59783 Answer

94(f) 92)83487 Answer

95. You and some friends have pitched in to buy a lottery ticket, because you are living in a TV movie and it seemed like a fun idea. It turns out to be a winning ticket. Ignoring any fractional part or remainder, give the following amounts that each friend (including yourself) receives, assuming that each person receives the same amount. Answer

95(a) 10 friends win $7419. Answer
95(b) 27 friends win $93↊0. Answer
95(c) 89 friends win $429↊3. Answer
95(d) 765 friends win $679↊925. Answer

96. You are a farmer trying to buy hay for your livestock. Answer

96(a) A ton of hay costs $827; but you only need 0;9 ton. How much does 0;9 ton cost? Answer
96(b) You've lost your records from the year before, but you know that you paid $6↊2;30 for 0;9 ton of hay. What was the price of hay per ton last year? Answer
96(c) It turns out you only need 0;7 ton, not 0;9 ton. How much will that cost, if you've been quoted a price of $827 for 0;9 ton? Answer

5.4.6.2 Division with Fractional Parts

Division with fractional parts is also a bit difficult, but it boils down to the same thing. There are two possible situations here; the first is when we have a fractional *dividend*, and the second when we have a fractional *divisor*.

A fractional *dividend* makes no difference to our operations; we simply continue adding digits to the quotient as if the dit weren't there. In our quotient, we transfer the dit to the same place as it sits in the dividend.

```
        ↊;9 3
    ---------
8 3 ) 7 4 ↊;3 9
```

(Obviously, we're not showing our work here.) Note that the answer here is identical to that we saw earlier[1↋], except for the dit (the *radix point*). All that is needed is to transfer

[1↋] *See supra*, starting at 7↋.

the radix point to the same location in the quotient that it occupies in the dividend; when our problem is set up in a tableau, as this is, that's a simple matter of putting it directly above itself. 74↊39 ÷ 83 = ↊93; 74↊;39 ÷ 83 = ↊;93.

A fractional divisor is a bit trickier. The trick is *to make it no longer fractional.* We do this simply by moving the dit enough places to the right that we are left with a whole number divisor. As long as we do the exact same thing to both the divisor *and* the dividend, we will get the right answer.

$$8;3\,\overline{)\,7\;4\;↊;3\;9}$$

Above, we see our familiar problem again, but with radix points placed in different places. We are stuck with an obnoxious fractional divisor, and we want to get rid of it. So we move the dit over one place:

$$8\;3;\overline{)\,7\;4\;↊;3\;9}$$

So now we have an integral divisor. But if we do the problem as it is, we'll get the wrong answer; we have to move the dit the same number of places in *both* the dividend *and* the divisor. So what we want is this:

$$8\;3\,\overline{)\,7\;4\;↊\;3;9}$$

(Since there's nothing after the dit, we can stop writing "83;" and just write "83.") Now we can do our division and get our answer:

$$\begin{array}{r} ↊\;9\;3 \\ 8\;3\,\overline{)\,7\;4\;↊\;3;9} \end{array}$$

(Again, without showing our work.) Now, we just saw that with fractional dividends all we need to do is put the dit in the same place in the quotient that it is in the dividend:

$$\begin{array}{r} ↊\;9;3 \\ 8\;3\,\overline{)\,7\;4\;↊\;3;9} \end{array}$$

And there we are. As you can see, the trick to doing these problems is to make them like multi-digit divisor problems by moving the radix point. Thus do we reduce two types of difficult problem to only one.

Exercises 5.15

97. Divide, to completion or to at least four fractional places. Answer

97(a) 4) 1 9 7 Answer **97(b)** 5) 3 7 6 4 Answer **97(c)** 6) 9 8;3 7 Answer

97(d) 4 5;7) 8 ↋ 4;6 7 Answer **97(e)** 5;7) ↋;5 3 2 8 Answer

97(f) ↋;7) ↋;5 3 2 8 Answer **97(g)** 1 4;3 2 7) ↊;7 ↊ Answer

98. You are ordering boxes of paper your company and need to know the price per ream. There are 12 reams in a box, and you've purchased 44 boxes. The total price was $↊6;↊2. How much did you pay per ream? Answer

99. You are a farmer who raises, among other things, rabbits as livestock. Over the last four weeks, your rabbits have gone from 56 to 1↊1. How many rabbits have you gained per day over that time? Answer

9↊. You are a teacher trying to calculate grades for a student. The student received a grade of ↊4 on his midterm and ↋4 on his final. What is his overall grade? Answer

5.4.7 Short Division

Fortunately, we don't always have to do long division; sometimes we can use a quicker method, unsurprisingly called *short division*.

Short division uses the same tableau format that long division uses; so we set up our problem like so:

$$8\,\overline{)\,8\;4\;5\;2}$$

Just as with long division, we collect digits to the left of the dividend until we have a number equal to or larger than the divisor; that is, we assemble enough of the leftmost digits in the dividend to acquire a *partial dividend*. Here, we only need one such digit. We then do the calculation, placing the first digit of our quotient above the partial dividend and the remainder below it.

$$\begin{array}{r} 1 \\ 8\,\overline{)\,8\;4\;5\;2} \\ 0 \end{array}$$

We follow this by using the remainder from our last step along with the next digit of the dividend to form another partial dividend; those digits are 0 and 4, forming the number 4. Our divisor does not go into 4; or, alternatively, it goes into 4 zero times. So we place the 0 in our quotient and the remainder, 4, in our lower line:

$$\begin{array}{r@{\,}l} & 1\ 0 \\ \cline{2-2} 8\big) & 8\ 4\ 5\ 2 \\ & 0\ 4 \end{array}$$

Do the same again, assembling a new partial dividend with the remainders from our last step. The partial dividend is 45. $45 \div 8$ is 6 R5, so we put the 6 into our quotient and the 5 below:

$$\begin{array}{r@{\,}l} & 1\ 0\ 6 \\ \cline{2-2} 8\big) & 8\ 4\ 5\ 2 \\ & 0\ 4\ 5 \end{array}$$

Then we again form a new partial dividend with the remainder from the previous step, 5, and the next digit of the full dividend, 2. That forms the number 52. $52 \div 8$ is 7 R6, so we place the 7 into our quotient and our 6 below the dividend:

$$\begin{array}{r@{\,}ll} & 1\ 0\ 6\ 7 & \quad \mathrm{R}\ 6 \\ \cline{2-2} 8\big) & 8\ 4\ 5\ 2 & \\ & 0\ 4\ 5\ 6 & \end{array}$$

We'll stop there, and see that $8452 \div 8 = 1067$, with a remainder of 6. We could continue in the same way and gather fractional digits by adding a dit and 0 to the right of the dividend, but for demonstrative purposes this will suffice.

Short division is also extremely useful for checking *divisibility*. If a number is evenly divisible by another, clearly the remainder of dividing the one from the other will be 0; e.g., if 548299↊38 is divisible by 7, then 548299↊38 % 7 = 0. If we perform this problem by short division, saving the remainders but ignoring the quotient, we can (relatively) quickly determine even divisibility.

$$\begin{array}{r@{\,}ll} & & \quad \mathrm{R}\ 0 \\ \cline{2-2} 7\big) & 5\ 4\ 8\ 2\ 9\ 9\ ↊\ 3\ 8 & \\ & 1\ 6\ 4\ 1\ 0\ 3\ 4\ 0 & \end{array}$$

Doing this, we may as well get the quotient along with the remainder; but the remainder is the primary object of the exercise. If the remainder of the problem is 0, then the dividend is evenly divisible by the divisor; if it is anything other than 0, then it is *not* evenly divisible by the divisor. This method provides a straightforward test for divisibility with those divisors which don't have easy, at-a-glance tests.

Exercises 5.16

9↋. Determine divisibility by 7 of the following numbers; use short division to do so. Answer

9↋(a) 4893. Answer **9↋(b)** ↊483. Answer **9↋(c)** 9↋537. Answer
9↋(d) 74↊↋8. Answer

↊0. Give the quotient for the following problems; use short division to do so. If not evenly divisible, find the remainder *and* fractional places up to 4 digits. Answer

↊0(a) $4\overline{)\,7\;4\;5\;8}$ Answer **↊0(b)** $7\overline{)\,9\;4\;5\;5}$ Answer

↊0(c) $5\overline{)\,↊\;6\;↋\;4\;7}$ Answer

5.4.8 Checking Multiplication and Division Results

We've already seen that addition and subtraction can be used to check the results of each other[20]; multiplication and division can check one another the same way, because the two operations are the inverses of one another.

As an example, take the following two problems:

$$\begin{array}{r} 8\;9\;4\;↋ \\ \times \qquad ↋ \\ \hline 8\;0\;7\;6\;1 \end{array} \qquad\qquad \begin{array}{r} 8\;0\;7\;6\;1 \\ \div \quad 8\;9\;4\;↋ \\ \hline ↋ \end{array}$$

Both the left and the right say essentially the same thing, but in inverse ways. So after completing a particularly complex or difficult multiplication or division problem, do the opposite operation and make sure that your answer is correct.

5.4.9 Factoring

We saw ways to determine many of the factors of a number before we started studying long division.[21] Unfortunately, however, most larger numbers have factors beyond 1–10, most of which we have seen are so easy to work with. The process of finding out what the factors of a number are is called *factoring*, and it is a central process in which division is used. Oftentimes, we'll be able to factor using the simple divisibility rules we learned

[20] *See supra*, Section 5.2.6, at 51. [21] *See supra*, Section 5.4.5, at 6↋.

above[22]; however, sometimes we'll have to go a bit farther. To do that, it's typically easiest to use a table.

Let's begin with a four-digit number; say, 67↊6. We put this number on the *top* of our *factoring table*.

67↊6	

In the left-hand column of this table, we will be placing our *factors*; in the right, we will be placing our *quotients*.

Let's start with the easy ones. This number is clearly *even*, so begin with 2:

67↊6	
2	

2 is our *factor*. Then, in the right-hand column, put down the *quotient*; that is, the quotient of the factor just put in the left column and the number above it:

67↊6	
2	33↋3

Now we see that 33↋3 is no longer even; as an odd number, it is *not* divisible by 2. So we have to move on to the next highest prime: 3.

67↊6	
2	33↋3
3	1139

33↋3 is clearly divisible by 3 (since it ends in 0, 3, 6, or 9), so we put a 3 in the left column; we then do 33↋3 $\div$ 3 = 1139, and put the quotient in the right column. 1139 is also divisible by 3 (since it ends in 0, 3, 6, or 9), so we do the same again:

67↊6	
2	33↋3
3	1139
3	453

[22] *See supra*, Table 5.1, at 70.

And again, we see that 453 is also divisible by 3; we continue on with 3 until the quotient is not itself divisible by 3:

67↊6

2	33Ɛ3
3	1139
3	453
3	159
3	5Ɛ

Now, 5Ɛ is plainly *not* divisible by 3; it does not end in 0, 3, 6, or 9. So we continue on with the next-highest prime: 5. Sadly, we don't have an easy test for dividing by 5, so we'll just have to try it out. 5Ɛ is *not* divisible by 5. So let's try the next-highest prime, 7. 5Ɛ is *not* divisible by 7 (we know this because of our *multiplication table*); so we continue on to our next-highest prime, Ɛ. 5Ɛ is *not* divisible by Ɛ (again, we know this from our multiplication table), so we move on to 11. We can continue on in this way; but as it happens, we've picked an unlucky number to factor, as 5Ɛ is itself a prime, and we'll have to try many numbers before we'll realize this unless we happen to know it.

Once we have primes in both columns, as we do here (3 and 5Ɛ), we divide our right-hand number by itself, which of course makes 1; we then place these into our table:

67↊6

2	33Ɛ3
3	1139
3	453
3	159
3	5Ɛ
5Ɛ	1

So we know that the prime factors of 67↊6 are 2, 3, 3, 3, 3, and 5Ɛ. We also know that every number in the right-hand column is a factor of 67↊6, as well, although we also know that none of them (save the last) is prime.

We further know that the *product* of any of the numbers in the left column is a factor. $3 \times 3 = 9$ is a good illustration here. We can tell just by looking at 67↊6 that it is divisible by 9; it meets the rules of divisibility by 9 from Table 5.1.[23] And the product of 3 and 3 is 9.

We can do as much experimenting as we wish, but let's do the ultimate experiment and find the product of all the prime factors by one: $2 \times 3 \times 3 \times 3 \times 3 = 116$. We can easily find that $116 \times 5↋ = 67↊6$, so clearly 116 is a factor. We can also easily see that $2 \times 3 = 6$ is a factor, because 67↊6 ends in a 6, and any number ending in 0 or 6 is divisible by 6.

So doing the problem this way, we identify at a glance all of the *prime factors* (the numbers in the left column); some of the *composite factors* (the numbers in the right column); and, by multiplying the prime factors in various combinations, the remaining factors of the number.

Exercises 5.17

↊1. Find all the factors, prime and otherwise, of the following numbers. Answer

↊1(a) 72 Answer **↊1(b)** 67 Answer **↊1(c)** 64 Answer **↊1(d)** 678 Answer
↊1(e) ↋69↊ Answer **↊1(f)** 499 Answer **↊1(g)** 22 Answer **↊1(h)** 343. Answer

5.5 Arithmetic on Vulgar Fractions

Vulgar fractions, you may recall, are fractions which are expressed as *ratios* of the number of parts present compared to the number of parts in a whole. We've seen that *digital fractions* give us an extremely easy way of doing arithmetic on fractional values; so easy, in fact, that for the most part there is no difference in handling fractional values than integral ones. This is the method which you're likely to use in most of your dealings with fractions.

However, we've also learned that digital fractions aren't capable of exactly expressing all values; some fractions simply don't fit into them. Such fractions include one fifth and one seventh; they take the form of infinitely repeating patterns, called *repeating fractions*. While often we can use a subset of these patterns and get a result that is close enough (e.g., using 0;2497 for one fifth), if we need perfectly exact results, we need to use *vulgar fractions*, which can exactly express any fractional value.

In this section, we will learn how to conduct the basic four functions on vulgar fractions. While the methods of doing so are more involved than our simple digital fraction algorithms, they are still needed for many purposes.

5.5.1 Basic Concepts for Working with Vulgar Fractions

Before we begin formally learning these methods, however, we need to explore a few basic concepts. It would probably be helpful to review the information in Section 1.5.1[24] to start off with; then we will examine a few new concepts we'll need to work in this branch

[23] *See supra*, at 70. [24] *See supra*, at 9.

of arithmetic. But first, some exercises to ensure that the material from Section 1.5.1 has been properly digested.

Exercises 5.18

72. Identify the numerator and the denominator of the following fractions. Answer

72(a) $\frac{4}{7}$ Answer **72(b)** $\frac{8}{9}$ Answer **72(c)** $\frac{7}{4}$ Answer **72(d)** $8\frac{3}{2}$ Answer

72(e) $\frac{134}{87}$ Answer

73. Tell whether the fraction is proper, improper, or mixed. Answer

73(a) $\frac{5}{8}$ Answer **73(b)** $\frac{8}{10}$ Answer **73(c)** $4\frac{12}{10}$ Answer **73(d)** $4\frac{87}{95}$ Answer

73(e) $\frac{87}{14}$ Answer

5.5.1.1 Greatest Common Factors

The *greatest common factor* is exactly what it sounds like: it is the greatest number which is a factor of two or more numbers.

greatest common factor

the highest number which is a factor of all of two or more numbers

The easiest way to find the greatest common factor is to simple factor both numbers, list them in order, and find the largest number that is in both lists. Take such a process with the numbers 10 and 24:

10	
2	6
2	3
3	1

24	
2	12
2	7
7	1

By the rules we learned about factoring,[25] we can see that the greatest common factor of 10 and 24 is 4.

However, there is an easier way, pioneered by the great Greek geometer Euclid. The greatest common factor of two numbers divides the difference of the two numbers; this may seem mysterious, but it is true. So to find the greatest common factor of two numbers, we can use the following algorithm:

- Divide the larger number (your dividend) by the smaller (your divisor), calculating the quotient and the remainder.

[25] *See supra*, Section 5.4.9, at 86.

- Make your last divisor into your new dividend, and make your last remainder your new divisor.
- Continue doing this until you arrive at a remainder of 0.
- Your last non-zero remainder will be your greatest common factor.

Let's try an example. What is the greatest common factor of 4762 and 98↊8?

4762) 98↊8 = 2 R 57↊4

57↊4) 4762 = 9 R 292

292) 57↊4 = 2 R 40

40) 292 = 8 R 12

12) 40 = 3 R 6

6) 12 = 6 R 2

2) 6 = 3 R 0

So we start with the larger number divided by the smaller, then turn the divisor of that problem into the dividend in a new one, with the remainder of that first problem serving as the new divisor We continue to do this until our remainder is 0, then we know that our last non-zero remainder is our greatest common factor. In this case, it is the surprisingly low 2.

Exercises 5.19

↊4. Find the greatest common factor of the following lists of numbers. Answer

↊4(a) 46, 36 Answer **↊4(b)** 47, ↊0 Answer **↊4(c)** ↊32, 849 Answer

↊4(d) 444, 778 Answer **↊4(e)** 57↊9, ↊843 Answer **↊4(f)** 68, 93↋4. Answer

↊4(g) 93, 9393 Answer

5.5.1.2 Least Common Multiple

Closely related to the greatest common factor is the *least common multiple*, which again is mostly self-definitive:

> ***least common multiple***
>
> the lowest multiple of two or more numbers; the lowest number which is a multiple of all of a list of two or more numbers

A *multiple* of a number is any number which can be obtained by repeatedly adding that number to itself. The reader will no doubt realize that this is essentially *multiplication*, hence the name "multiple."

So the least common multiple is the lowest number which is a multiple of all of a list of two or more numbers. How do we determine this?

Most simply, we know that multiplying two numbers together gives us *a* common multiple, by necessity. This leads us to a necessary conclusion: the least common multiple of two numbers cannot possibly be greater than the product of those numbers. So, to find the largest possible least common multiple of two numbers, simply multiply them:

$$68 \times 84 = 4768$$

If we're seeking the least common multiple of 68 and 84, we know that it cannot possibly be higher than 4768.

Whether or not there are lower common multiples, however, is a trickier question. We can, of course, simply proceed by trial and error:

	2	3	4	5	6	7
68	114	180	228	294	340	378
84	148	210	294	358	420	474

It should be relatively clear what we've done here. We've set up our two numbers on the left, in green; we want to find the least common multiple of them. So we line up on the top, in red, the first few integers and just start multiplying; $68 \times 2 = 114$, $68 \times 3 = 180$, and so on. We do this for both numbers, then compare the results, looking for an identical value in both rows. We find 294, which we now know to be the least common multiple of 68 and 84.

While 294 is quite a bit lower than 4768, and therefore preferable, the process of finding the least common multiple this way is tedious and error-prone. It is very easy, when making a table like this and multiplying repeatedly, to make a mistake, which could cause one to miss the common multiple one's looking for. Further, in this case, the least common multiple came at 4 and 5; what if they had come at much higher numbers? Our table could quickly become prohibitively large.

Fortunately, there is a shorter method, which involves finding the *greatest common factor*.[26] Using "lcm" as an abbreviation for *least common multiple* and "gcf" as an abbreviation for *greatest common factor*, the equation for finding the least common multiple is as follows:

$$\mathrm{lcm}(a, b) = \frac{|a \cdot b|}{\mathrm{gcf}(a, b)}$$

This is a lot of fancy notation; but what it means is that there is a simple algorithm for finding the least common multiple of two numbers:

1. Multiply the two numbers.
2. If the result of the last step is negative, make it positive. (Simply remove the "−".)
3. Find the greatest common factor of the two numbers.
4. Divide the result of step 2 by the result of step 3.

[26] *See supra*, Section 5.5.1.1, at 87.

Let's try this three-step algorithm on a real problem: finding the least common multiple of 48 and 8↊.

1. $48 \times 8↊ = 3528$.
2. The last step is already positive, so no need to change anything. Still at 3528.
3. The greatest common factor of 48 and 8↊ is 2.
4. $3528 \div 2 = 1874$, which is the least common multiple.

As we can see, the trickiest part of this is simply remembering the steps; the work itself is child's play, stuff we have done many times before.

Exercises 5.1↊

↊5. Find the least common multiple of the following pairs. Answer

↊5(a) 43, 879 Answer **↊5(b)** 5↊, ↊7 Answer **↊5(c)** 8, 20 Answer
↊5(d) 20, 64 Answer **↊5(e)** 6, 7 Answer **↊5(f)** 6, 17 Answer
↊5(g) 4789, ↊↋↋↊ Answer

5.5.1.3 Converting Fractions

We frequently find ourself in need of *converting* fractions. Converting a fraction means taking an improper fraction and making it a mixed number, or vice versa. The process for doing this is simple.

To convert an improper fraction into a mixed number, divide the numerator by the denominator; the quotient is the whole number, and the remainder is the numerator of the mixed number. Let's take an example:

$$\frac{47}{8}$$

The numerator is in blue and the denominator is in green. So we simply divide the numerator by the denominator:

$$47 \div 8 = 6 \text{ R}7$$

We keep the same denominator as before; the quotient is our new whole number, and the remainder is our new numerator.

$$\frac{47}{8} = 6\frac{7}{8}$$

And it's that simple. Practice for a bit; but it won't take much. You're already very familiar with all these operations, and are merely combining them in a different way.

Exercises 5.1↋

↊6. Convert the following into mixed numbers. Answer

ᘔ6(a) $\frac{18}{6}$ Answer **ᘔ6(b)** $\frac{26}{4}$ Answer **ᘔ6(c)** $\frac{87}{ᘔ}$ Answer **ᘔ6(d)** $\frac{14}{3}$ Answer
ᘔ6(e) $\frac{19}{4}$ Answer

Converting mixed numbers to improper fractions is even simpler: just multiply the whole number by the denominator, add to it the numerator of the mixed number, and retain the same denominator. For example:

$$6\frac{7}{8}$$

We know that $6 \times 8 = 40$, and $40 + 7 = 47$, so we can do the following:

$$6\frac{7}{8} = \frac{47}{8}$$

The whole number part of a mixed number means that we have that many wholes in addition to the *fractional* part, and the denominator means that we have that many parts per whole. So we need to multiply the number of wholes by the number of parts per whole, then add the number of parts less than one whole, to get the total number of parts.

Exercises 5.20

ᘔ7. Convert the following to improper fractions. Answer
ᘔ7(a) $7\frac{8}{9}$ Answer **ᘔ7(b)** $4\frac{3}{4}$ Answer **ᘔ7(c)** Ɛ$\frac{ᘔ}{14}$ Answer **ᘔ7(d)** $2\frac{1}{2}$ Answer
ᘔ7(e) $3\frac{2}{3}$ Answer

5.5.1.4 Vulgar Fractions as Division Problems

Fundamentally, vulgar fractions are simply division problems. We use the two concepts to express the same underlying reality in different circumstances, which is why we often treat vulgar fractions as an entirely independent thing. However, underneath it all, they are the same.

This identity of the two concepts is easiest to see in improper fractions. So let's look at one now:

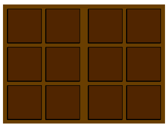
$$\frac{14}{10}$$

We've already seen that we can convert this to a mixed number by dividing the numerator by the denominator, where the quotient is our whole number and the remainder is the numerator of our remaining fraction.[27] But why is that the case?

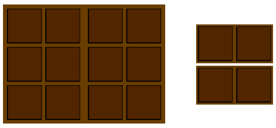
Because the numerator becomes our *dividend* and the denominator becomes our *divisor*. One could even say that when we do this, we're not converting an improper fraction into a mixed number; we're simply doing a division problem and getting an answer.

[27] *See supra*, Section 5.5.1.3, at 91.

That's why one of the notations for division is identical to that of vulgar fractions, as we saw on page 63. Considered as a vulgar fraction, the $\frac{14}{10}$ we saw above means that we have wholes consisting of 10 parts, and we have 14 parts in total. The following chocolate bar is 1:



This, clearly, is made up of 10 equal parts. That means that the following is $\frac{14}{10}$:



This, just as clearly, is 14 equal parts. We know that 1 whole chocolate bar is 10 of these parts; and we also know that we have 14 of these parts. So we have a vulgar fraction:

$$\frac{14}{10}$$

Now, we saw in Section 1.5.2 that digital fractions are just vulgar fractions where we know the denominator without having to write it; that is, that it's just 10, possibly with extra zeroes added on depending on how long the digital fraction is. We also know that $\frac{14}{10}$ is equal to $1\frac{4}{10}$, thanks to our work in Section 5.5.1.3. That means that $\frac{14}{10}$ is the same as 1;4. Let's prove it, though; let's assume that vulgar fractions and division are the same by dividing 14 by 10.

$$\begin{array}{rr} & 1;4 \\ 10 & \overline{)\,14;0} \\ & 10 \\ & 4\,0 \\ & 4\,0 \\ & 0 \end{array}$$

And there we have it. Notice that we could have stopped here at the dit, and had a quotient of 1 and a remainder of 4. That would give us the mixed number $1\frac{4}{10}$. Instead, we chose to go on, and wound up with the digital fraction 1;4.[28] We know already, from our understanding of what digital fractions are, that this is the correct answer.

[28] Also notice that we could have divided simply by sliding the dit to the left one place, since this problem is division by a 1 followed by some number of zeroes; but that would defeat the purpose of this exercise.

We can do this, of course, with any vulgar fraction. Take our now-familiar example $\frac{47}{8}$:

$$\frac{47}{8} = 6\frac{7}{8} = 6;↊6$$

It may be helpful to do the division problem—that is, $\frac{47}{8}$, or $47 \div 8$—if this concept isn't yet clear to you. Just remember that, with a vulgar fraction, the *numerator* is the *dividend* while the *denominator* is the *divisor*:

$$\begin{array}{r} ;↊\ 6 \\ 8\overline{)\ 7;0\ 0} \\ 6\ 8 \\ 4\ 0 \\ 4\ 0 \\ 0 \end{array}$$

Here, we've left out the whole number part; we know that there are 6 wholes here, and can add it in later. And we see that this is a straightforward application of the long division algorithms that we studied so painstakingly in Section 5.4.6, nothing more.

Division and vulgar fractions are the same thing, two different ways of expressing identical concepts. This is an important notion to understand; we will see why in a moment, when we begin to reduce fractions.[29]

Table 5.2 on page 95 lists some of the most common digital expansions of vulgar fractions; its contents are best committed to memory. Notice that many equal the same thing; e.g., $\frac{1}{2}$ is the same as $\frac{2}{4}$ is the same as $\frac{4}{8}$ is the same as $\frac{8}{14}$; we'll learn why in the next section.

5.5.1.5 Reduction of Fractions

Finally, there is the notion of *reduction*. *Reduction* of a fraction is an important concept that we will often use; it simplifies almost all operations regarding fractions.

> ***reduction***
>
> of fractions, the process of expressing a fractional value with the lowest possible numerator and denominator

The meaning of this might not be readily apparent; so let's review it now.

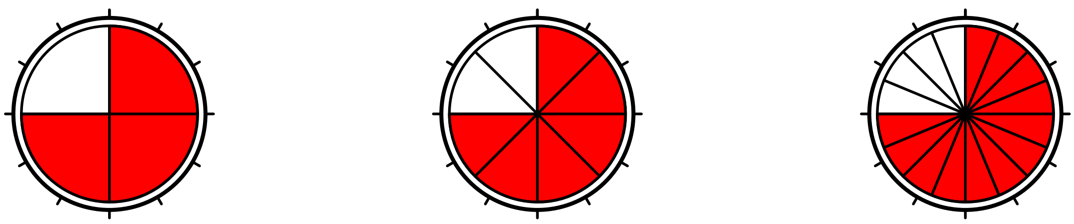
Now, it should come as no surprise to hear that multiple division problems can yield the same answer. For example, $↊ \div 5$ and $12 \div 7$ both equal 2. Since vulgar fractions and division are the same, then, it will come as no surprise that multiple fractions can likewise yield the same answer; that is, many different fractions have the same value.

[29] *See infra*, Section 5.5.1.5, at 94.

Vulg. Frac.	Dig. Frac.	Vulg. Frac.	Dig. Frac.	Vulg. Frac.	Dig. Frac.
1/2	0;6	1/3	0;4	2/3	0;8
1/4	0;3	2/4	0;6	3/4	0;9
1/6	0;2	2/6	0;4	3/6	0;6
4/6	0;8	5/6	0;↊		
1/8	0;16	2/8	0;3	3/8	0;46
4/8	0;6	5/8	0;76	6/8	0;9
7/8	0;↊6				
1/14	0;09	2/14	0;16	3/14	0;23
4/14	0;3	5/14	0;39	6/14	0;46
7/14	0;53	8/14	0;6	9/14	0;69
↊/14	0;76	↋/14	0;83	10/14	0;9
11/14	0;99	12/14	0;↊6	13/14	0;↋3

Table 5.2: Digital Expansions of Common Vulgar Fractions

If we doubt this, it might help to visualize several fractions to observe their equality physically:



$\frac{3}{4}$ $\frac{6}{8}$ $\frac{10}{14}$

We clearly see that, though we've taken a whole and divided it into differently-sized parts, the same *amount* of each whole is filled in red. That's because each of these vulgar fractions, when divided, has the same quotient; that is, $3 \div 4 = 0;9$, $6 \div 8 = 0;9$, and $10 \div 14 = 0;9$. So while they *appear* different, each has the same value, because when the numerator is divided by the denominator, each has the same quotient.

We have also seen, when we discussed division of numbers with fractional parts,[2↊] that we can move the radix point of the divisor to the right as long as we also move the radix point of the dividend to the right the same number of places, adding zeroes if necessary. Finally, we've learned that moving the radix point to the right is the same as multiplying by a one followed by some zeroes.[2↋]

Before we proceed to examine the implications of these facts, let's summarize them briefly here:

[2↊] *See supra*, Section 5.4.6.2, at 82. [2↋] *See supra*, 5.3.3, at 56.

- Vulgar fractions and division are the same thing.
- Multiple vulgar fractions can have the same value, just as multiple division problems can have the same quotient.
- We can adjust division problems by moving the radix point around, so long as we move it the same number of places in both the dividend and the divisor.

That last point is very important; because it turns out that we can not only move the radix point around, we can do *anything* to the problem, as long as we do the same thing to both the numerator and the denominator. Try it; for example, take a random division problem and make an arbitrary adjustment to both operands:

$$\text{↊}\,\overline{)\,8\;4} \qquad\qquad 2\;6\,\overline{)\,2\;1\;0}$$

The first problem is 84 ÷ ↊; the second problem is 210 ÷ 26, in which both the dividend (numerator) and the divisor (denominator) have been multiplied by 3. What is the answer?

$$\begin{array}{r} \text{↊} \\ \text{↊}\,\overline{)\,8\;4} \end{array} \qquad\qquad \begin{array}{r} \text{↊} \\ 2\;6\,\overline{)\,2\;1\;0} \end{array}$$

They both equal the same thing. We can do anything we like to the problem and still get the same answer, provided that we do the same thing to both operands.

Sometimes, the parts of a vulgar fraction are too large to be easily worked with. Let's take an example:

$$\frac{49}{93}$$

Doing arithmetic on smaller numbers is always easier than on larger ones; so is there any way to reduce the complexity of this vulgar fraction by making the numbers smaller?

Yes, *provided that we do so in the same way to both the numerator and the denominator*. Since we want to make the numbers smaller, we'll rely on division, rather than multiplication. How do we know by what number we should divide them?

Well, we want to make the numbers as small as possible, so we want to divide them by the largest number we can; and we need to divide them both by the same number. So we want to divide them by the largest number which evenly divides both of them. You guessed it: we want to divide them both by their *greatest common factor*.

The greatest common factor of 49 and 93 is 3; so we divide both the numerator and the denominator by 3 to come up with a more manageable fraction:

$$\frac{49}{93} \div \frac{3}{3} = \frac{17}{31}$$

$\frac{17}{31}$ is much more manageable, size-wise, than $\frac{49}{93}$, and it has the exact same value. So we have *reduced* our fraction.

Incidentally, this equation also shows us why this concept—manipulating vulgar fractions by performing the same operation on both parts—works. We're dividing both the numerator and the denominator by 3; this is functionally the same as dividing the whole fraction by $\frac{3}{3}$. But a number divided by itself is equal to 1; so $\frac{3}{3} = 1$. And any number divided by 1 equals itself; so dividing this number by $\frac{3}{3}$ results in the same number. Clearly, then, we're not changing the value of $\frac{49}{93}$ when we reduce it to $\frac{17}{31}$; we're just making it easier to deal with. The whole can be written as follows:

$$\left(\frac{49}{93} \div 1\right) = \left(\frac{49}{93} \div \frac{3}{3}\right) = \frac{17}{31}$$

And all is still right with the world.

Exercises 5.21

↊8. Reduce the following fractions. Answer

↊8(a) $\frac{49}{96}$ Answer **↊8(b)** $\frac{54}{88}$ Answer **↊8(c)** $\frac{46}{70}$ Answer **↊8(d)** $\frac{16}{30}$ Answer

↊8(e) $\frac{39}{↋8}$ Answer **↊8(f)** $\frac{12}{5↊}$ Answer **↊8(g)** $\frac{16}{69}$ Answer **↊8(h)** $\frac{47}{↊1}$ Answer

↊8(i) $\frac{↋4}{7↊8}$ Answer **↊8(j)** $\frac{7}{19}$ Answer

5.5.2 Addition and Subtraction

Unlike with the four functions on normal numbers, *addition* and *subtraction* on vulgar fractions is actually trickier than *multiplication* and *division*; so unlike the last section, our work will be getting easier as we go on.

At first glance, addition of vulgar fractions seems simple, even trivial once we've learned addition of normal numbers. Take a basic example:

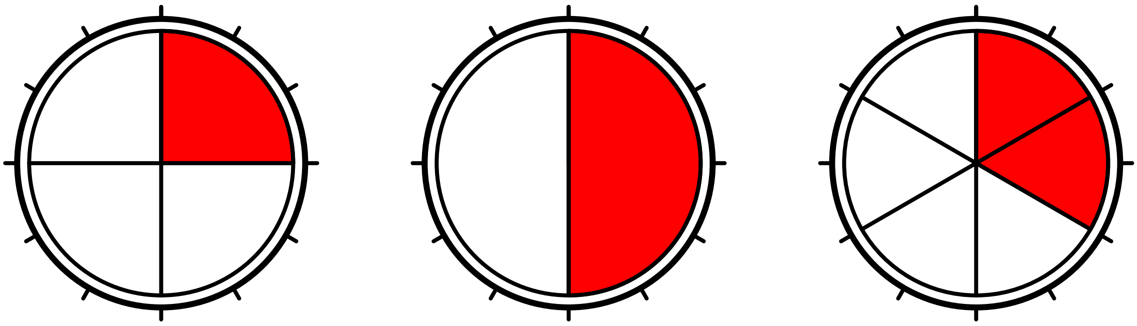
$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

Easy! Just add the numerators! And indeed, in this case that policy works; $\frac{3}{4}$ is the correct answer. But what if we have a different problem:

$$\frac{1}{4} + \frac{1}{2} = ?$$

Can we simply add the numerators? If so, which denominator will be in the solution? Or do we add the denominators, too?

Let's see what happens when we attempt this:



$$\frac{1}{4} \quad + \quad \frac{1}{2} \quad = \quad \frac{2}{6}$$

No, that clearly doesn't work. We'd expect the fraction of the answer to have *more* red filled in, not *less*; we'd expect it to have an amount of red filled in equal to the amount of red in both the addends put together. So we need a different algorithm.

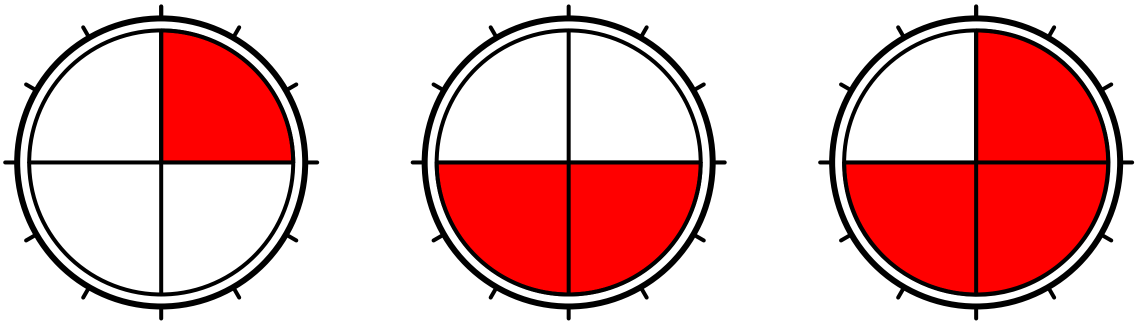
The answer comes to us when we remember *least common multiples*. Remember that vulgar fractions can be *reduced*: as long as we do the same thing to both the numerator and the denominator, we'll have the same value but with different figures. Similarly, we can *multiply* the denominators to make them both the same, doing the same multiplication to the numerators.

In this case, we have the denominators 4 and 2. It's easy enough to see that their least common multiple is 4. One of the denominators is already 4, so we don't need to do anything to that. The other denominator is 2; we can multiply it by 2 to make it 4, then multiply the numerator by 2, as well, making it 2. So we end up with $\frac{2}{4}$.

$$\frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$$

We know that $\frac{1}{2}$ and $\frac{2}{4}$ are the same value, because we only multiplied by $\frac{2}{2}$, which is the same as $2 \div 2$, which is equal to 1, and any number multiplied by 1 is itself. So clearly $\frac{1}{2} = \frac{2}{4}$.

Now that the numerators and denominators are the same, we can proceed to add the fractions and see if this method works:



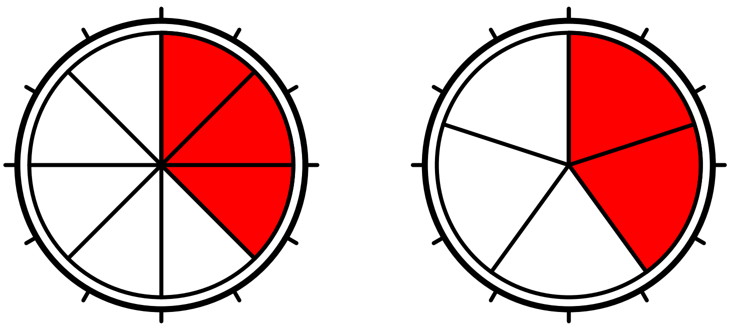
$$\frac{1}{4} \quad + \quad \frac{2}{4} \quad = \quad \frac{3}{4}$$

That, clearly, is correct.

So the process of addition of vulgar fractions is:

- If the denominators are the same, simply add the numerators.
- If the denominators are different, find their least common multiple.
- Multiply both the numerator and denominator of one or both fractions to make the denominators equal their least common multiple.
- Proceed to add the fractions.

Let's try a more complicated example:



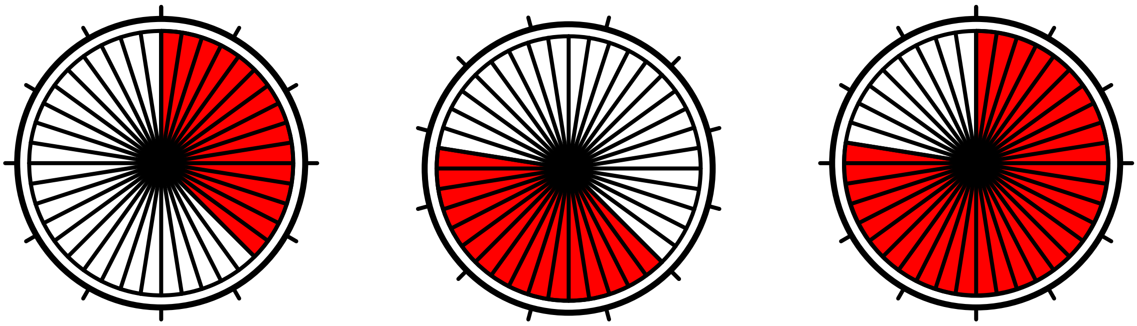
$$\frac{3}{8} \quad + \quad \frac{2}{5} \quad = \quad ?$$

The denominators are not the same, so we must find their least common multiple. The least common multiple of 8 and 5 is 34. Now we need to render both fractions such that their denominators are 34. To do this, we need the quotients of 34 and each fraction's denominator. So we do $34 \div 8 = 5$ and $34 \div 5 = 8$. Then we do the following equations to get our new fractions:

$$\frac{3}{8} \times \frac{5}{5} = \frac{13}{34}$$

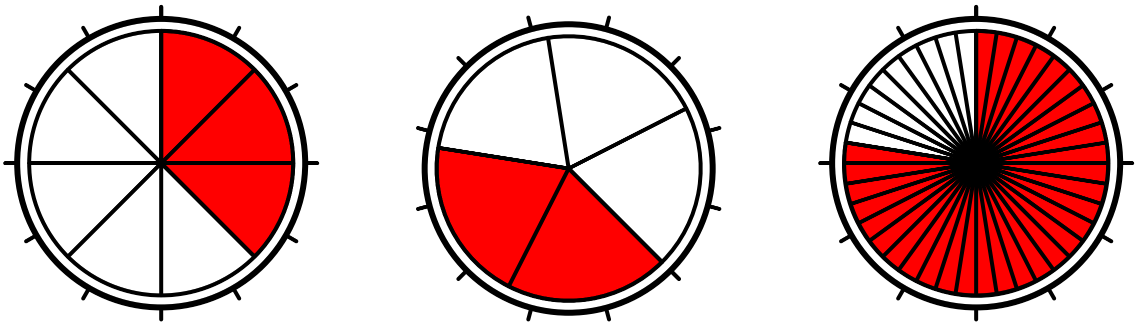
$$\frac{2}{5} \times \frac{8}{8} = \frac{14}{34}$$

Now, we are ready to do our actual addition:



$$\frac{13}{34} \quad + \quad \frac{14}{34} \quad = \quad \frac{27}{34}$$

Let's look at the original fractions and see if these quantities ar ethe same:



$$\frac{3}{8} \quad + \quad \frac{2}{5} \quad = \quad \frac{27}{34}$$

And that looks about right, visually confirming what we already knew to be correct through the arithmetic.

Remember that a number divided by itself is 1, and a number divided by 1 is itself. So if you have a whole number in a fraction problem, put it on top of a 1 to do your calculations:

$$\left(\frac{14}{26} + 3\right) = \left(\frac{14}{26} + \frac{3}{1}\right)$$

Subtraction works identically, except that once we have given both fractions equal denominators, we *subtract* rather than add. There are no surprises there.

Exercises 5.22

↊9. $\frac{3}{7} + \frac{4}{9}$ Answer **↊↊.** $\frac{4}{5} + \frac{1}{14}$ Answer **↊↋.** $\frac{2}{3} + \frac{3}{4}$ Answer **↋0.** $\frac{2}{8} + \frac{1}{4}$ Answer

↋1. $\frac{7}{14} + \frac{2}{17}$ Answer **↋2.** $\frac{2}{↋} + \frac{7}{1↊}$ Answer **↋3.** $\frac{1}{4} + \frac{3}{4}$ Answer **↋4.** $\frac{2}{14} + \frac{1}{8}$ Answer

↋5. $\frac{4}{29} - \frac{21}{54}$ Answer **↋6.** $\frac{↊}{↋} - \frac{4}{↊}$ Answer **↋7.** $\frac{3}{4} - \frac{1}{4}$ Answer **↋8.** $\frac{2}{14} - \frac{1}{8}$ Answer

↋9. You and two friends have purchased two pizzas, each of which was cut into eight slices. If one of your friends has three slices and the other five, how much of the pizza have they eaten? Answer

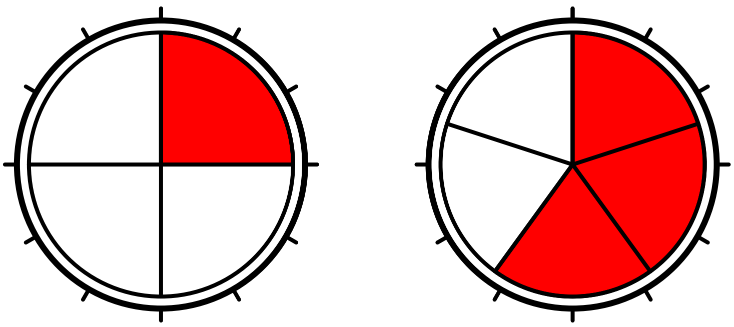
↋↊. In the last question, how much pizza is left for you to eat? Answer

↋↋. You are managing an office supply warehouse and, at the beginning of the week, had 340 reams of paper in stock. At the end of the week, you had 216 reams in stock. What fraction of your stock did you ship? Answer

5.5.3 Multiplication and Division

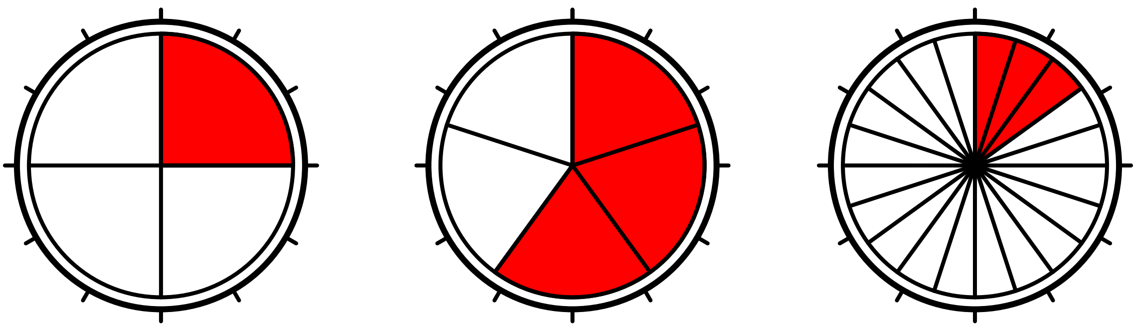
Multiplication of fractions is easy, much easier than addition and subtraction of them. It's easier because it does not involve finding the least common multiple and equalizing the denominators before you can begin your actual operation. Instead, you simply multiply the *numerators* by the *numerators* and the *denominators* by the *denominators*, and that's the end of the matter.

Let's look at the fractions visually, as we did above with our addition problems.



$$\frac{1}{4} \quad \times \quad \frac{3}{5} \quad = \quad ?$$

Now let's try just multiplying the *numerators* and *denominators*:



$$\frac{1}{4} \quad \times \quad \frac{3}{5} \quad = \quad \frac{3}{18}$$

At first glance, this doesn't look right; multiplication produces *larger* numbers, not *smaller* ones, doesn't it? But this algorithm results in a smaller number; so it might be wrong.

However, we need to think a little more about what's really happening here before we decide that. Note that the smaller the number we multiply by, the smaller the increase in size of the product; let's take a look:

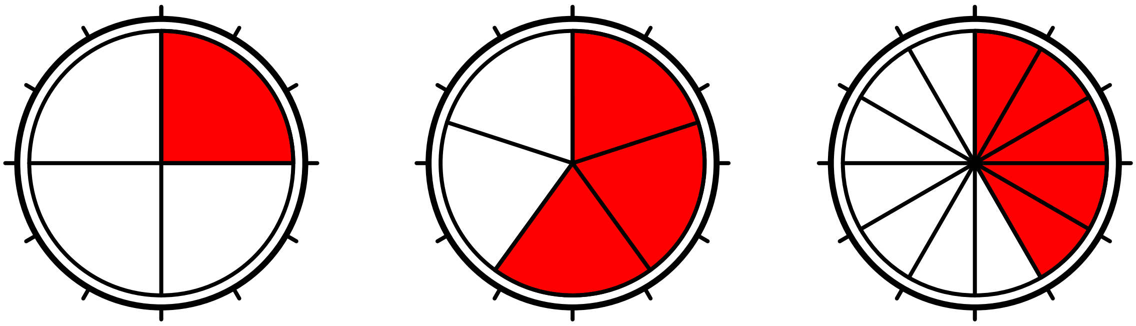
$$\begin{aligned} 3 &\times 4 = 10 \\ 3 &\times 3 = 9 \\ 3 &\times 2 = 6 \\ 3 &\times 1 = 3 \end{aligned}$$

As the *multiplier* decreases, so also does the *product*; that much seems simple enough. But what do we do when the multiplier is *less than one?*

Clearly, if we continue this pattern, the product will continue to decrease as the multiplier decreases, even if that means that the product will be less than the multiplicand. So multiplying by a fraction will actually result in a product *smaller* than the multiplicand.

So $\frac{1}{4} \times \frac{3}{5}$ really does equal $\frac{3}{18}$; and multipling fractions really is that easy.

Now, we can move on to division, which is nearly as simple. When dividing vulgar fractions, we see the opposite occurrence as when multiplying them; that is, while the *divisor* decreases, the *quotient increases*. So dividing by a fraction actually makes the quotient *larger* than the dividend, not smaller. Again, let's look at a visual representation:



$$\frac{1}{4} \quad \div \quad \frac{3}{5} \quad = \quad \frac{5}{10}$$

Notice that our quotient is larger than our dividend. The reader is encouraged to construct a table similar to that we did above for multiplication if this concept is hard to understand.

The algorithm for dividing vulgar fractions is simple: multiply by the *reciprocal*. The reciprocal of a number, remember, is that number which, when multiplied by it, gives a product of 1; so the reciprocal of 4 is $\frac{1}{4}$. Remembering that any number divided by 1 is itself, and that vulgar fractions are just division considered in a different way, we can do the following:

$$4 = \frac{4}{1}$$

$$\frac{4}{1} \times \frac{1}{4} = \frac{4}{4} = 1$$

So $4 \times \frac{1}{4} = 1$; that means that $\frac{1}{4}$ is the reciprocal of 4.

And we can extend this: the reciprocal of a number is 1 divided by that number; e.g., as we saw above, the reciprocal of 4 is $1 \div 4$.

That's how it works for whole numbers; how about the reciprocals of vulgar fractions?

Well, the reciprocal is that number which, when our original number is multiplied by it, produces a product of 1. In vulgar fractions, any number which is both the numerator and denominator (that is, any number divided by itself) is equal to 1. So we need to find the vulgar fraction which, when multiplied by our original vulgar fraction, results in an equal numerator and denominator.

The answer to this is always the same fraction, but with the numerator and the denominator flipped. Let's take a random fraction as an example:

$$\frac{4}{7}$$

Now let's flip it, multiply, and look at our result:

$$\frac{4}{7} \times \frac{7}{4} = \frac{24}{24} = 1$$

A little thought will show that this works every time, even with whole numbers, as seen above. So we can reliably and easily extract the reciprocal from any vulgar fraction: simply switch the numerator and denominator.

Having learned that, we're ready to learn the rule for division of such fractions.

Division is simply multiplication by the reciprocal of the divisor; so in our visual problem above,

$$\left(\frac{1}{4} \div \frac{3}{5}\right) = \left(\frac{1}{4} \times \frac{5}{3}\right) = \frac{5}{10}$$

And there you have it: an easy and direct method for performing division on vulgar fractions. Division having formerly been by far our most difficult operation, it's refreshing to have it be so simple here.

Exercises 5.23

100. $\frac{4}{7} \times \frac{8}{9}$ Answer **101.** $\frac{1}{2} \times \frac{4}{8}$ Answer **102.** $\frac{3}{4} \times \frac{2}{3}$ Answer **103.** $\frac{3}{4} \times \frac{2}{3}$ Answer

104. $\frac{3}{4} \div \frac{↊}{↋}$ Answer **105.** $\frac{1}{2} \div \frac{7}{4}$ Answer **106.** $\frac{14}{17} \div \frac{24}{3}$ Answer **107.** $\frac{4}{3} \div \frac{12}{17}$ Answer

108. Answer the following questions. Answer
 108(a) What is $\frac{1}{3}$ of $\frac{1}{4}$? Answer
 108(b) What is 0;9 of $\frac{1}{3}$? Answer

109. You are Robin Hood, and have stolen $34\frac{1}{3}$ Maz of gold from the rich in order to give it to the poor. You're no haphazard benefactor, however; you've selected the very poorest families to receive your largesse. Solve the following with fractions. Answer
 109(a) How much will each of 8 families receive? Answer
 109(b) How much will each of 4 families receive? Answer
 109(c) How much will each of 1↊ families receive? Answer

10↊. You are managing an office supply warehouse, and you are told by your floor manager that a large order for staples came in from an insurance office; he further tells you that he had to ship three quarters ($\frac{3}{4}$) of your staple stock to fill the order. You are aware that prior to the order you had 48 boxes of staples in stock. How many boxes of staples do you have left? Answer

10↋. You are planning a wedding and are expecting 134 guests (based on their responses to your invitations). Your food budget is \$6500. Your current choice for meals involves a price of \$46 per person ("a head," in the vulgar phrase). If you choose this option, what portion of your food budget is taken up with dinner, and what portion is left for cake? Answer

5.6 Order of Operations

Sometimes we have many operations to do at once, rather than just the one that we have seen so far. In other words, the problems we have seen thus far have been like $3 + 4$; sometimes, however, we will see $3 \times 4 - 6 \div 9$. How do we know which operations to do first?

The answer to that is the *order of operations*, a simple system which tells us which operations to do in what order. The default is that they go in order from left to right; but this order can often be changed.

1. Operations of the same type go from left to right. E.g., $3 + 4 + 7$ should be done $3 + 4 = 7$, then $7 + 7 = 12$.

2. Multiplication.

3. Division.

4. Addition.
5. Subtraction.

There is a convenient mnemonic to remember the order of operations:

Multiplication	**D**ivision	**A**ddition	**S**ubtraction
My	**D**ear	**A**unt	**S**ally

"My Dear Aunt Sally" is an easy way to remember something that's very important when doing arithmetic. Operations of the same type are done from left to right; if the operations are of different types, they are done in "My Dear Aunt Sally" order.

Let's take an example:

$$3 + 4 + 7 + 9 + 8 + \text{Ƨ}$$

This problem has five operations, and they are all of the same type. So we do them in a simple order, from left to right. First, $3 + 4$:

$$7 + 7 + 9 + 8 + \text{Ƨ}$$

Then, $7 + 7$:

$$12 + 9 + 8 + \text{Ƨ}$$

Then, $12 + 9$:

$$\text{1Ɛ} + 8 + \text{Ƨ}$$

Then, $\text{1Ɛ} + 8$:

$$27 + \text{Ƨ}$$

And, finally, $27 + ↊$:

$$35$$

So $3 + 4 + 7 + 9 + 8 + ↊ = 35$. Of course, addition is *associative*, so it really doesn't matter in what order we do the operations. But what if we had a subtraction problem?

$$57 - 4 - 7 - 9 - 8 - ↊$$

Subtraction is *not* associative, so it's quite important that we do it in the correct order if we want the right answer. So first, we have to do $57 - 4$:

$$53 - 7 - 9 - 8 - ↊$$

Then, $53 - 7$:

$$48 - 9 - 8 - ↊$$

Then, $48 - 9$:

$$3↋ - 8 - ↊$$

Then, $3↋ - 8$:

$$33 - ↊$$

Then, finally, $33 - ↊$:

$$25$$

If we did things in a different order, we'd get the wrong answer; so knowing that these must be accomplished left-to-right is very important.

Then, of course, there are those circumstances where we have multiple different operations:

$$57 - 4 \div 7 + 9 \times 8 \times \text{↊}$$

Since we have multiple different operations, we have to remember "My Dear Aunt Sally." First, we do *multiplication*; but we note that there are *two* multiplication problems. What to do? We simply return to the default order for them: we do all multiplications first, but we do them in order from left to right. So first, 9×8:

$$57 - 4 \div 7 + 60 \times \text{↊}$$

$9 \times 8 = 60$; then, we do $60 \times \text{↊}$:

$$57 - 4 \div 7 + 500$$

Now, we've finished all the multiplication operations, so we move on to *division*. There is only one of these, so we do $4 \div 7$; this doesn't come out evenly, so we'll truncate our answer to four places:

$$57 - 0;6\text{↊}35 + 500$$

Now that we've finished all the division problems, we move on to *addition*, and do $0;6\text{↊}35 + 500$:

$$57 - 500;6\text{↊}35$$

And having finished addition, we now do the *subtraction*:

$$-465;6↊35$$

And we know that $57 - 4 \div 7 + 9 \times 8 \times ↊ = -465;6↊35$. Order of operations was critical in determining the proper answer to this problem.

It's also important to note that when you're *writing* an equation, you need to keep the order of operations in mind. If you don't write it such that the proper operations will be done first, you'll get the wrong answer. Please keep this in mind, particularly when developing your answers to word problems.

There are a few further complication to order of operations (namely, parentheses and exponentiation), but for our purposes at this time, you've learned enough to test your knowledge and move on.

Exercises 5.24

110. $4 + 4 - 3$ Answer **111.** $4 - 4 + 3$ Answer **112.** $4 \times 4 + 3$ Answer
113. $4 + 4 \times 3$ Answer **114.** $4 \times 4 \div 2$ Answer **115.** $4 \div 4 \times 2$ Answer
116. $7 + 3 \times 4 \div 2$ Answer **117.** $7 \times 4 + 8 \div 6$ Answer

5.7 Casting Out Elvs

We've seen several ways of checking each operation that we've learned[30]; however, there is one method which works for all of the basic four functions: *casting out elvs*.

"Elv" here is short for "eleven"; and before we get into the details of the method, we need to discuss the concept of the *digit sum*:

digit sum
the sum of the digits of a number

We've already seen this in our test for divisibility by ↋ (in Table 5.1 at 70); here, we see the full usefulness of the procedure. It turns out that, when taking a digit sum, any ↋ can simply be ignored without changing the result; so when we're doing it, we simply "cast off" the ↋s, which gives the process its name.

Furthermore, in any operation, *the digit sum of the result equals the same operation done of the digit sums of the operands.* Let's examine this statement by example. Take an addition problem:

[30] *See supra*, Section 5.2.6, at 51; and Section 5.4.8, at 86.

$$\begin{array}{r} 6\,\text{Ɛ}\,8\,4 \\ +\,4\,5\,\text{Ɛ}\,\text{Ɛ} \\ \hline \text{Ɛ}\,5\,8\,3 \end{array}$$

We know how to do this quite well by this point; and we also know that we can check the result by subtracting one of the addends from the sum and seeing whether the difference is equal to the other addend. E.g., we can do Ɛ583 − 45ƐƐ = 6Ɛ84; and by doing this, we know that we have the right answer.

However, let's try something else: let's take the digit sums of both addends and the sum:

$$\begin{array}{r} 6\,\text{Ɛ}\,8\,4 \\ +\,4\,5\,\text{Ɛ}\,\text{Ɛ} \\ \hline \text{Ɛ}\,5\,8\,3 \end{array} \qquad \begin{array}{r} 7 \\ +\,9 \\ \hline 1\,4 \end{array}$$

Remember that, when we're taking the digit sums, we can ignore all the Ɛs. So we get the digit sum of the first addend by adding $6 + 8 + 4 = 12 + 4 = 16$, then $1 + 6 = 7$. Then we take the digit sum of the second addend by adding $4 + 5 = 9$. Then we take the digit sum of the sum by doing $5 + 8 + 3 = 11 + 3 = 14$, and though we could go one step further and say $1 + 4 = 5$, we'll stop there.

Notice that the digit sums of the addends sum to the digit sum of the sum. That is, $7 + 9 = 14$.

Obviously, this will work backwards for subtraction, as well. Let's try some multiplication, though:

$$\begin{array}{r} 6\,\text{Ɛ}\,8\,4 \\ \times\qquad 5\,\text{Ɛ} \\ \hline 3\,5\,3\,2\,3\,8 \end{array} \qquad \begin{array}{r} 7 \\ \times\,5 \\ \hline 2 \end{array}$$

At first glance, this looks like it doesn't work; after all, $7 \times 5 \neq 2$. However, $7 \times 5 = 2\text{Ɛ}$; and remember that we can ignore all our Ɛs. So really, this *does* work; and we do have the right answer.

Finally, let's try this on a division problem:

$$\begin{array}{r} 3\,5\,3\,2\,3\,8 \\ \div\qquad 5\,\text{Ɛ} \\ \hline 6\,\text{Ɛ}\,8\,4 \end{array} \qquad \begin{array}{r} 2 \\ \div\,5 \\ \hline 7 \end{array}$$

Note that, for as for multiplication, we have to add an Ɛ to the dividend in order to make this work. But of course, we can; the presence or absence of an Ɛ makes no difference. Here, our check is 2Ɛ ÷ 5 = 7, which is, of course, true.

This is often a *much* faster way of checking our arithmetic than the full algorithms we've discussed in previous sections; make use of this method when you can.

Chapter 6
Advanced Arithmetic

Arithmetic is built upon the basic four functions, but can quickly move beyond them into more advanced and interesting things. For example, we will see that numbers consisting of lower numbers multiplied repeatedly by themselves have lots of special properties when we study exponentiation and roots[1]; we'll see special ways of using exponentiation and roots when we study logarithms[2]; and we'll learn about when some digits in a number might be more important than others.[3]

6.1 Estimating Numbers

Estimation is an important skill when doing arithmetic; it is the ability to know where one can stop determining exact results, and to know how inaccurate an answer one has obtained after doing so. This, in turn, enables one to know whether the inaccuracy is acceptable, or too significant to trust the answer. First we will examine the concepts of *rounding* and *truncation*; then we will look at the concept of *significant digits*, which is extremely important in scientific pursuits.

6.1.1 Rounding and Truncation

We've observed, in the last chapter, how to get exact answers to arithmetic problems to any number of digits. However, often we don't really need that much accuracy; we just need to get an *approximate* answer. Often we can do arithmetic on rounder numbers without even putting pen to paper; but if there are too many digits, we won't be able to juggle them all correctly in our heads. When we *estimate* the number instead, we don't get an exact result; but we may well be able to get one that's close enough for our purposes.

The easiest way to do this approximation is by *truncation*:

> ***truncation***
>
> the act of approximating a number's value by selecting a place and simply cutting off the remaining places, without other adjustment to the number

We *truncate* when we have a string of digits that we don't need, and we'd like to shorten the number for easier handling. Consider the following:

$$0;2497\,2497\,2497\,24$$

This is the digital fraction expansion of the vulgar fraction $\frac{1}{5}$; specifically, the first 12 digits of it. It's very unlikely that we need this much precision for whatever we need this

[1] *See* Section 6.2 on ↋3. [2] *See* Section 6.2.3 at 109. [3] *See* Section 6.1.2 at ↋0.

value for. Let's assume that we'd be happy with just the first four digits, as four dozenal digits is almost certainly sufficient for our purposes. We *truncate* the expansion to the number of digits we want just by lopping off anything past the fourth digit:

$$0;2497$$

It's that simple. Of course, truncation can't be applied in the same way to digits to the *left* of the dit; each digit going that direction will change the number's value by too much for the truncation to be useful. But to the *right* of the dit, we can easily cut off digits, because each digit farther to the right has a smaller and smaller value.

If we need to approximate a number to the *left* of the dit, we can use *rounding*:

> ***round***
>
> to round is the act of approximating a number's value by selecting a place and adjusting it according to its following place, replacing all subsequent digits with zeroes

When we round a number, we aren't simply lopping off extra digits, or converting them to zeroes; we're changing the number to be a bit closer to the true value while still sacrificing some accuracy. Take our same digital fraction expansion of $\frac{1}{5}$:

$$0;2497\,2497\,2497\,24$$

This, again, is far more accuracy than we're ever likely to need; two digits is a much more realistic (and convenient) degree of precision. If we were to *truncate* to two digits, we'd get:

$$0;24 \text{ (truncated)}$$

But we only need to look at the third digit of the original number to see that this is quite inaccurate; the following digit is a 9, which means that the true value is much closer to 0;25 than it is to 0;24. So instead, we can *round* the number:

$$0;25 \text{ (rounded)}$$

This is much closer to the true value.

When we round, we look at the value of the digit one place farther to the right than that to which we are rounding. If that digit is *less than six*, then we are closer to the current value of our rounding place than to the next number up, so we leave the rounding place alone. If that digit is *six or more*, then we're closer to the next number up, so we adjust the rounding place by adding 1 to it.

To demonstrate: round the values 0;254 and 0;258 to two digits.

$$0;254 \rightarrow 0;25 \qquad\qquad 0;258 \rightarrow 0;26$$

To the left, because the *third* digit is 4 (less than 6), a *second* digit of 5 is closer to the true value than a second digit of 6 would be. So we *round down* and leave the second digit (the one we're actually rounding to) as 5.

To the right, because the *third* digit is 8 (6 or greater), a *second* digit of 6 is closer to the true value than a second digit of 5 would be. So we *round up* and make the second digit (the one we're actually rounding to) a 6.

So much for rounding digits to the *right* of the radix point (dit). But we can also round them to the *left* of the dit, on basically the same principle. Take the following number:

$$43\,ᘔ973;3987\,Ɛ43$$

A large number with a lot of precision. But perhaps we want to do a quick mental calculation to get an approximate answer, so we don't need all those digits of precision. Let's say we want to round this number to the nearest quadqua.

First, we identify the digit just to the *right* of the digit to which we wish to round. The quadqua digit is the 3, and the digit to the right is the ᘔ. Next, we determine whether that digit to the right is either *less than 6* or *6 or more*. Here, it is ᘔ, so it is *6 or more*. That means we round *up*, making the 3 into a 4. Now we don't need to know any of the digits to the *right* of the digit to which we are rounding; that is, after all, the whole point of the exercise. So we get this:

$$43\,ᘔ973;3987\,Ɛ43 \longrightarrow 44\,0000$$

Note that we are *not* writing the dit or the digits to the right of it; that is because they, too, have become zeroes, and consequently have no impact on the value of the number. And so we have rounded to the nearest quadqua.

Please further note that sometimes rounding can *cascade*; that is, it can require action beyond the digit you're rounding to. Take the following example:

$$6ƐƐ\,8905$$

To round to the nearest quadqua: first, we look at the digit in the quadqua place, which is the first Ɛ. Next, we look at the digit immediately to the right of that, which is 8. Clearly, then, we must round *up*. But rounding up means we have to raise the digit we're rounding to by 1; but Ɛ + 1 = 10, which means we've gone beyond where one digit place can take us. So we need to raise the *next digit to the left* by 1, and make the current digit a 0. But the next digit to the left is *also* Ɛ. So the rounding cascades to the left until the digits no longer overflow their place.

$$6ƐƐ\,8905 \longrightarrow 700\,0000$$

If we run into the far left of the number, and still must add a digit, we can simply make that final digit into two digits; e.g., ƐƐ 8905 would become 100 0000. And these rules are sufficient to round any number, as necessary.

Exercises 6.1

1. Truncate the following numbers to four digits. Answer
1(a) 5;43892 Answer **1(b)** 834;399ᘔ3 Answer **1(c)** 93;578297 Answer
2. Round the following numbers to four digits. Answer
2(a) 5;43892 Answer **2(b)** 834;399ᘔ3 Answer **2(c)** 93;578297 Answer
3. Round to the nearest triqua. Answer
3(a) 592ᘔᘔ Answer **3(b)** 597ᘔ2 Answer **3(c)** Ɛ6329 Answer
3(d) 9Ɛ64ᘔ Answer

6.1.2 Significant Digits

Rounding and truncation are the ways we shorten numbers to a certain estimation; that is, to reflect a certain degree of accuracy. *Significant digits* are how we determine whether that degree of accuracy is correct, or at least acceptable, according to the problem we're currently doing.

For example, if we weigh a bag of potatoes and find that it weighs 1 Mag, and we know that it contains 7 potatoes, the weight of each potato can be determined to be 1 ÷ 7, or 0;186ᘔ 3518 6ᘔ35 . . . , and on forever if we want. But this isn't really accurate; that's much more accuracy than our scale, which only gave us 1 digit, makes possible. So we have to *round* to the nearest significant digit; our best answer for the weight of each potato, then, is 0;2 Mag. Writing more than that is pretending to more accuracy than we really have.

Significant digits are the way we resolve this problem:

> ***significant digit***
>
> those digits in a number which carry meaning according to the resolution currently available; often also called significant figures

By learning how to count significant digits, we learn how to ensure that we are writing our numbers to the greatest accuracy we have a right to, *and no more*. We also ensure that we're not inferring more information from a number we see than we have a right to; that is, when we see a number, we'll know how accurate (or inaccurate) it really is.

There are five simple rules for counting significant digits, only one of which is at all confusing:

1. All non-zero digits are significant.
2. Zeroes in between non-zero digits are significant.
3. Leading zeroes (zeroes at the left end of a number) are never significant.

4. In a number with a radix point (a dit), trailing zeroes (zeroes at the right end of a number) are significant.

5. In a number without a radix point, trailing zeroes are sometimes significant and sometimes not.

Applying the first four of these rules is basically trivial. Let's look at a few examples below. We will color significant digits in blue, and non-significant digits in black.

0;00572 — Rule 1 (all non-zero digits are significant) shows that 572 is significant, and Rule 3 (leading zeroes are never significant) shows that the first 0;00 is *not* significant.

↊07;9300 — Rule 1 (all non-zero digits are significant) shows that ↊, 7, 9, and 3 are significant; Rule 2 (zeroes between non-zero digits are significant) shows that the first 0 is significant; and Rule 4 (trailing zeroes in a number with a radix point are significant) shows that the final 00 is significant. So all these digits are significant.

Rules 1–4 are, clearly, easily applied. Rule 5 is somewhat trickier. Essentially, because leading zeroes on the left of the number are always present merely to show scale, and not as values themselves, we know that these are never significant (Rule 3). However, trailing zeroes at the right of the number (assuming that there is no radix point) may be present merely to show the scale of the number, or may be significant values. So a closer look is required.

The significance of trailing zeroes cannot be determined from the number alone; it must be determined by context. For example, if the number is the product of multiplication, and both the factors of that product have only two significant digits, then the result cannot have more than two significant digits, so trailing zeroes in excess of that cannot be significant.

One important thing to remember about significant digits is that only measured quantities care about them. That is, we know that π is a number with an infinite number of places; we don't worry about significant digits with it. But if we *measure* a quantity and do arithmetic on it, we need to pay attention to the significant digits throughout the process.

Significant digits of a sum or difference are determined by the leftmost significant digit of the addends, subtrahend, or minuend. For example:

$$\begin{array}{r} 103;7286 \\ +\ 430;4 \\ \hline 533;\text{↋}286 \end{array}$$

For multiplication and division, though, the answer has the same number of significant digits as the factor, dividend, or divisor with the *least* number of significant digits. For example:

$$\begin{array}{r} 103;7286 \\ \times\ 430;4 \\ \hline 44379;864\text{↊} \end{array}$$

Here, the factor with the least number of significant digits is 430;4, which has 4 significant digits. That means that our answer really has only four significant digits, shown above highlighted in blue. It follows, then, that we should really write our answer thus:

$$44380$$

(Note that we've removed the digits after the radix point; they're not significant, but writing them to the right of the dit would make them appear so. Also, we have rounded up appropriately.)

Incidentally, this serves as an excellent example of determining when trailing zeroes are significant. Here, we have one trailing zero to the left of the radix point; this the type of zero that Rule 5 makes ambiguous. Here, we know that it is *not* significant, because we know what the factors were.

It seems like we're losing significant precision here, but we're really not; we're just carefully maintaining the precision that we started with. Fortunately, this only applies to *measured quantities*; when we're dealing just with numbers, we needn't think about it at all.

Exercises 6.2

4. Give the number of significant digits in the following numbers. If you cannot answer fully, give the best answer you can and explain why your answer is incomplete. Answer

4(a) 34 Answer **4(b)** 183;4 Answer **4(c)** 0;05890 Answer

4(d) 330010 Answer

5. You're building a rectangular compost bin, and you want it to be 4;3 Grafut long by 2;7 Grafut deep (that being your available space). You want to calculate the area this will yield you, so remembering that area is length multiplied by width, you multiple 4;3 by 2;7 and get ↊;↋9. Is this really your answer? Answer

6. You are designing a model bridge which needs to hold 0;64 Maz minimum (that being the mass of the model train that will traverse it). Your other calculations indicate that your bridge will hold 0;65 Maz; however, you only have one significant digit justified by your measurements. Can you rely on your bridge? Answer

6.2 Exponentiation and Roots

Exponentiation and roots are the process whereby we can do *repeated multiplication* (*exponentiation*) or *repeated division* (*roots*) by the same number. It comes with a special notation, but is really just an application of the simple multiplication and division rules we've encountered so much before.

While *exponentiation* and *roots* are just two sides of the same coin, like multiplication and division, the methods by which we calculate and manipulate them are different enough that we'll be considering them separately. We will start with exponentiation.

6.2.1 Exponentiation

Exponentiation is the process by which we raise a *base* to an *exponent* and produce a *power*; this is a mathematical way of describing what we're actually doing:

exponentiation

an operation describing multiplication of a base by itself a certain number of times, with the number of times superscripted after the base

The special notation that we use for exponentiation is what typographers call a *superscript*: a smaller number typeset slightly above the baseline, like so:

$$b^x$$

The x there is a *superscript*; that is the sign that we are doing exponentiation.

The different parts of an exponentiation operation are called the *base*, the *exponent*, and the *power*; below, the *base* is colored in blue, the *exponent* is colored in red, and the *power* is colored in green:

$$b^x = y$$

This is pronounced as "b raised to the x is y," "y is the xth power of b," or simply "b to the x is y."

What this actually means is that we are multiplying b by itself $x - 1$ times, giving an answer of y. So we can formally define the parts of an exponentiation problem as follows:

base

the number being repeatedly multiplied by itself in an exponentiation operation; the b in b^x

exponent

the number of times the base should be multiplied by itself (minus 1) in an exponentiation problem; the x in b^x.

power

the answer to an exponeniation problem; the y in $b^x = y$

As a demonstration, let's consider 5^4:

$$5^4$$

In this exponentiation problem, 5 is the *base* and 4 is the *exponent*. So our task is to multiply 5 by itself $4 - 1$ times; that is, 3 times:

$$5^4 = 5 \times 5 \times 5 \times 5$$

Note that, although we are multiplying 5 by itself $4 - 1$ times, since we're starting with 5 we still have four 5s in the equation. With that, we can compute our result:

$$5^4 = 21 \times 5 \times 5$$

$$5^4 = ↊5 \times 5$$

$$5^4 = 441$$

Some exponents are so common and important that they have special names; specifically, 2 and 3. The power of a number raised to an exponent of 2 is that number's *square*:

square

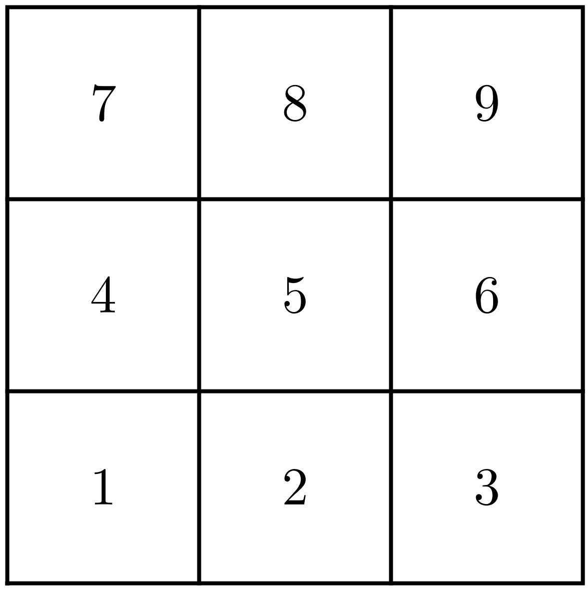
the power of a number raised to an exponent of two; the y in $b^2 = y$

So we say that the *square* of 3 is 9, because $3^2 = 3 \times 3 = 9$, and the *square* of 6 is 30, because $6^2 = 6 \times 6 = 30$. Often, we will describe such an equation as "six squared."

Square is also, of course, the name of a *shape*, and here we see one of the many intersections between geometry and arithmetic. The second power of a number is called its *square* because, geometrically, the second power of the length of a line describes the area of a *square* with sides of precisely that length.

Let's consider a line 3 units long:

We know that the *square* of 3 is 9, because $3^2 = 3 \times 3 = 9$. So let's experiment and draw a *square* on this line segment. Since it's 3 units long, its vertical side must also be 3 units long:



Clearly, taking the *square* of this line segment has given us a physical *square* with that same line segment as one of its sides. This is true for taking the square of any number.

Therefore, the *second power* (that is, any number raised to an exponent of 2) is called that number's *square*, and performing that operation on a number is called *squaring* it.

The same thing applies in three dimensions to the *third* power of a number, or its *cube*:

cube

the power of a number raised to an exponent of three; the y in $b^3 = y$

Instead of a square with a side equal to the number, cubing gives us a *cube* with a side equal to the number. Cubing is also a fairly frequent operation, but not nearly as common as squaring.

There are a couple of exponents that carry some unintuitive results, however, and so we need to examine them on their own. These exponents are 0 and 1.

Rule of Exponential Identity

Any number raised to the power of 1 is itself.

This is relatively straightforward. $b^1 = b$, because b multiplied by itself $1 - 1$ times is simply b. This is true for all values for b.

Rule of Zero Exponent

Any number raised to the power of 0 is 1, unless the base is 0.

This is the rule which is non-intuitive, and the reasoning behind it is quite deep, delving into number and set theory. It is further complicated by only being valid for *non-zero* bases; a base of 0 may equal 1, or it may equal something else, depending on the mathematician or the context in which it is being used.

For students, it is best to consider 0^0 equal to 1, though they should be aware that there is some controversy on the issue.

For non-zero bases, $b^0 = 1$, every time.

Exercises 6.3

7. 4^2 Answer **8.** 3^7 Answer **9.** 8^3 Answer **↊.** 2^3 Answer
↋. 2^8 Answer **10.** 6^8 Answer **11.** 7^4 Answer **12.** 2^{14} Answer
13. Give the squares of the following numbers. Answer
13(a) 5 Answer **13(b)** 27 Answer **13(c)** ↊8↋ Answer **13(d)** 82 Answer
14. Find the cubes of the following numbers. Answer
14(a) 8 Answer **14(b)** ↊8 Answer **14(c)** 2 Answer **14(d)** 18 Answer
15. Raise 38 to the fourth power. Answer

6.2.1.1 Arithmetic on Exponents

Exponentiation often makes arithmetic on large numbers considerably easier (a notion we'll see come into its own when we study logarithms[4]), due to a few simple relations.

Exponent Product Rule

$$b^m \cdot b^n = b^{m+n}$$

$$a^m \cdot b^m = (a \cdot b)^m$$

The exponential product rule is really two rules; we'll address each of those in turn. The first is the rule for different *exponents* with the same *base*:

[4] *See infra*, Section 6.2.3, at 109.

$$b^m \cdot b^n = b^{(m+n)}$$

In the above equation, m and n stand for exponents, while b stands for the base. Knowing this, it should be more or less self-explanatory: the product of two powers of a given base is equal to that base raised by the sum of the two exponents.

In other words:

$$4^8 = 4^4 \cdot 4^4$$

You can test this, as well, if you'd prefer; $4^8 = 31Ɛ14$, and $4^4 \cdot 4^4 = 194 \cdot 194 = 31Ɛ14$. This is true for any positive exponent.

The second Exponent Product Rule is similar, but applies in different circumstances; that is, it applies when the *bases* are different but the *exponents* are the same.

$$a^m \cdot b^m = (a \cdot b)^m$$

In this equation, a and b are the different bases, and m is the identical exponent. The formula shows us that the product of two bases each raised to the same exponent is the same as the product of those two bases together raised to the same exponent.

$$3^4 \cdot 5^4 = (3 \cdot 5)^4$$

It's easy to test: $(3 \cdot 5)^4 = 13^4 = 25369$, and $3^4 \cdot 5^4 = 69 \cdot 441 = 25369$. This is also true for any positive exponent.

The Exponent Product Rule also gives rise to the Exponent Quotient Rule:

Exponent Quotient Rule

$$b^m / b^n = b^{m-n}$$

$$a^m / b^m = (a/b)^m$$

Again, we really have two rules here. The first is that the quotient of a base raised to one exponent and the same base raised to a different exponent is equal to that same base raised to the exponent of the difference of the two bases.

$$b^{m-n} = b^m \div b^n$$

And we can test this rule pretty easily:

$$4^3 = 4^8 \div 4^5$$

$$54 = 31Ɛ14 \div 714$$

$$54 = 54$$

The second of the Exponent Quotient Rules is that the quotient of two different bases each raised to the same exponent is equal to the quotient of the two bases together raised to that same exponent.

$$(a/b)^m = a^m/b^m$$

Testing this rule is still equally easy:

$$4^3/2^3 = (4/2)^3$$

$$54/8 = 2^3$$

$$8 = 8$$

These relationships mean that we can easily do arithmetic on exponents simply by manipulating the exponents and not worrying about the bases or their values until we get to the final step.

Finally, there is the Exponent Power Rule:

Exponent Power Rule

$$(b^m)^n = b^{(m \cdot n)}$$

The Exponent Power Rule shows us that, when one power is raised by another exponent, that is the same as raising the original base by the product of the two exponents.

The Exponent Power Rule provides an excellent way of making long exponentiation problems shorter and easier. Take, for example, a rather large exponentiation problem:

$$2^{30}$$

This would require us multiplying 2 by itself *three dozen times*; this is doable, but time-consuming and error-prone. We can make this significantly simpler, however, and more importantly significantly shorter, by applying the Exponent Power Rule.

We know the first few powers of 2, of course: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 14$, $2^5 = 28$, $2^6 = 54$, $2^7 =$ ↊8, $2^8 = 194$, and so on. Let's note that $30 = 6 \times 6$. Thanks to the Exponent Power Rule, we know that

$$2^{30} = (2^6)^6 = ((2^6)^3)^2$$

Now, instead of three dozen operations, we need to perform only four: $2^6 = 54$, so we need to do $54 \times 54 \times 54$, then multiply the result of that by itself:

$$((2^6)^3)^2$$

$$(54^3)^2$$

$$10\,7854^2$$

$$113\,9\text{↊}01\,\text{↋}854$$

A quick check on a calculator shows us that 2^{30} does indeed equal 113 9↊01 ↋854. The problem still isn't what we'd call *fun*; but it's a whole lot easier than doing three dozen operations, and we owe it to the Exponent Power Rule.

We can simplify it even more by taking advantage of the Exponent Product Rule at the same time.

1. $2^2 = 4$
2. $(2^2)^2 = 2^4 = 14$
3. $(2^4)^2 = 2^8 = 194$
4. $(2^8)^2 = 2^{14} = 31\text{↋}14$
5. $(2^{14})^2 = 2^{28} = 9\text{↋↊}461594$
6. $(2^{28})(2^4) = 11397\text{↊}01\text{↋}854$

We were able to accomplish this without even knowing the powers of 2, as we had to in the previous method. In steps 1–5, we rapidly built ourselves up to larger powers of 2 using the Exponent Power Rule. In step 5, we reached a point at which adding our current exponent (28) to one of our low ones (4) would equal our desired exponent (30). We then used the Exponent Product Rule ($b^{m+n} = b^m \cdot b^n$) to simply multiply those two numbers, giving us the power 2^{30} as our answer.

Once again, such problems aren't really *fun*; but they are much less tedious and error-prone using the exponent rules than multiplying manually.

These rules give us a means to do quick arithmetic on what could otherwise be large and complicated problems. They work for all integral exponents, including negative exponents.[5]

Exercises 6.4

16. $4^3 \cdot 4^2$ Answer **17.** $3^3 \cdot 3^5$ Answer **18.** $6^3 \cdot 6^6$ Answer **19.** $6^7 \div 6^7$ Answer
1↊. $8^8 \div 8^5$ Answer **1↋.** $4^{12} \div 4^9$ Answer **20.** $(5^3)^4$ Answer **21.** $(2^3)^5$ Answer
22. $4^3 \cdot 3^3$ Answer **23.** $5^3 \cdot 5^3$ Answer **24.** $7^4 \cdot 4^4$ Answer **25.** $8^4/4^4$ Answer
26. $9^4 \div 3^4$ Answer

[5] *See supra*, Section 6.2.1.2, at ↋↊.

6.2.1.2 Negative Exponents

Negative exponents seem counterintuitive. After all, how can you multiply a number by itself a negative number of times? We're presented at first glance with the same problem as with zero for an exponent; it's genuinely difficult to figure out how to handle them.

The trick is that *negative numbers* are the opposite (the inverse) of *positive numbers*; so rather than doing *multiplication*, we do *division*. In other words, rather than *multiplying* the number by itself a given number of times, we *divide* that number by itself a given number of times.

Of course, dividing a number by itself simply equals 1, even if we do it forever; so first we divide 1 by the number, then divide that by the base *exponent*−1 times.

The astute reader will have noticed, then, that an exponent of −1 will then give the *reciprocal* of the base; that is, one divided by the base. That is:

$$b^{-1} = \frac{1}{b}$$

When the exponent is even lower than −1, we simply do the division repeatedly, which is the same as dividing by the power:

$$b^{-3} = \frac{1}{b \cdot b \cdot b}$$

An even easier and more direct notation is:

$$b^{-m} = \frac{1}{b^m}$$

So knowing how to do exponentiation and division is sufficient to know how to do negative exponents. Simply calculate the power as though the exponent were positive, then take its reciprocal (by dividing it from 1).

There is also the matter of negative *bases*; these, however, work as one would expect. Simply follow the rules for multiplying negative numbers; namely, that like signs multiply into positive numbers and unlike signs multiply into negative numbers. This means that *even* exponents of negative numbers will result in *positive* powers, while *odd* exponents of negative numbers will result in *negative* powers.

Exercises 6.5

27. 8^{-1} Answer **28.** 8^{-3} Answer **29.** 10^{-4} Answer **2↊.** 6^{-8} Answer
2↋. $8^{-7} \cdot 8^8$ Answer **30.** $3^3 \div 3^{-4}$ Answer **31.** $(8^{-2})^3$ Answer **32.** $(4^{-3})^{-3}$ Answer

6.2.1.3 Order of Operations

We have already learned the basic order of operations for doing arithmetic[6]; we learned that "My Dear Aunt Sally" will remind us to do our operations from left to right, but to

[6] *See supra*, Section 5.6, at page ↊2.

do first multiplication; then division; then addition; and finally subtraction. But where does *exponentiation* fit in?

Exponentiation	**M**ultiplication	**D**ivision	**A**ddition	**S**ubtraction
Excuse	**M**y	**D**ear	**A**unt	**S**ally

Dear Aunt Sally, unfortunately, has become a bit weird, and requires a few apologies every now and then. So now simply remember to *Excuse* My Dear Aunt Sally, and do your exponentiation problems first.

Finally, always do *parentheses* first.

parenthesis

a symbol for grouping mathematical operations in an equation; operations contained in parentheses are always done first, and multiple operations within parentheses are themselves done according to the normal order of operations

We're all familiar with parentheses in text (they're these curvy things we use to insert asides), but in mathematics they do something different. The order of operations (Excuse My Dear Aunt Sally) tells us what order to do various operations in; if we want to do them in a different order, we can group them in parentheses.

Parentheses are always done first. Perform any operations that are contained in parentheses prior to any other operations. For example:

$$8 - 4 + 2$$

Normal order of operations rules tells us to perform the addition first, then the subtraction, like so:

$$8 - 4 + 2$$
$$8 - 6$$
$$2$$

If for some reason we want the opposite order, we can simply use parentheses:

$$(8 - 4) + 2$$
$$4 + 2$$
$$6$$

Clearly, we get a very different answer here, due to the fact that subtraction is not commutative; so parentheses can be very important.

If there are multiple operations in a single set of parentheses, do them in the normal order of operations; if there are multiple sets of parentheses, do them left to right. If your parentheses are *nested* (that is, you have parentheses within parentheses, possibly several deep), do the innermost set first, then the next innermost, and so on until you've made it through.

And so we must modify our Aunt Sally mnemonic one more time, to take into account parentheses. So let's not just just ask people to excuse her oddness; let's ask them to *please* excuse it.

Parentheses	**E**xponentiation	**M**ultiplication	**D**ivision	**A**ddition	**S**ubtraction
Please	**E**xcuse	**M**y	**D**ear	**A**unt	**S**ally

Exercises 6.6

33. $3^3 \cdot 4$ Answer **34.** $3 \cdot 4^3$ Answer **35.** $(3 \cdot 4)^3$ Answer **36.** $4 \cdot 3^4 + 2$ Answer
37. $4 \cdot (3^4 + 2)$ Answer **38.** $2 \cdot 4 - 8^3$ Answer **39.** $2 \cdot (4 - 8^3)$ Answer

6.2.1.4 Compound Interest

Unlike the four functions we now know so well, exponentiation doesn't have immediately obvious applications. However, it's extremely important for a variety of mathematical disciplines (the most obvious is the hugely important calculus) and practical works. The most obvious of these is compound interest.

One of the perennial interests of man has been how to apply compound interest, and exponentiation provides a good and clear way to do so. The concept of compound interest is simple: each year a certain portion (say, 10%) of a deposit (say, $1000) is added back to that deposit. For one year, it's easy; 10% of $1000 is $100, so we have a new deposit amount of $1100. However, the second year we need 10% of $1100; and the next year there will be a new amount; and so on, as long as the deposit exists. How can we simply calculate amounts of compound interest without having to calculate each year's individual deposit amount?

Using *exponents*. Let's begin by expressing in an equation the interest and deposit on the first year; for this example, assume that 10% is expressed, in digital form, as 0;1[7]:

$$1000 + 1000(0;1)$$

Remember that *juxtaposition* means *multiplication*; so 1000(0;1) means 1000 × 0;10. This equation means the initial deposit of $1000 (the first 1000), plus the result of $1000 multiplied by 0;1 (the value of 10% of $1000). We do multiplication first in our order of operations, so we can calculate this amount easily:

[7] *See infra*, Section 6.3.4, at 126.

$$1000 + 1000(0;1)$$
$$1000 + 100$$
$$1100$$

A little thought will reveal that writing the equation with two "1000" is unnecessary; we can actually write $1000(1 + 0;1)$ with the same meaning. (The 1 assures that we're not getting *just* the interest, but the interest *and* the original $1000.) Like so:

$$1000(1 + 0;1)$$
$$1000(1;1)$$
$$1100$$

And we see that we get the same answer; that gives us our first year.

But when we have a constantly increasing base number, and that number is constantly increasing by the same value, we have what we call *exponential* change. That means that we can use exponentiation to find the answer. We simply multiply our base (the deposit) by the rate of increase $(1 + 0;1)$ *raised to the exponent equal to the number of years in question.* To start with, let's look at our total deposit at the end of the third year:

$$1000(1 + 0;1)^3$$
$$1000(1;1)^3$$
$$1000(1;331)$$
$$1331$$

Without exponentiation, we'd have to do it the slow way, calculating each year individually:

$$1000(1 + 0;1)$$
$$1000(1;1)$$
$$1100$$
$$1100(1 + 0;1)$$
$$1100(1;1)$$
$$1210$$
$$1210(1 + 0;1)$$
$$1210(1;1)$$
$$1331$$

Clearly, exponentiation makes these problems substantially quicker and easier. Of course, we can use *any* rate of interest just by changing the 0;1; we merely chose this because there are twelve months in the year. We can get the compounded interest for any number of years just be changing the exponent.

Exercises 6.7

3↊. Assume an initial deposit of $1 2000 in a bank account which gathers 3% interest each month. For this exercise, assume that 0;03 is the digital version of 3%.[8] Answer

3↊(a) After six months, what will be the value of the account? Answer

3↊(b) After six months, how much additional money does the investor have? Answer

6.2.1.5 "Exponentially"

Our discussion of compound interest does reveal the proper usage of a word that, in popular culture, we often hear misused: "exponentially".

Most of the time, people use this term merely to mean "a lot." For example, "immigration has increased exponentially in the last ten years," or "our taxes increased exponentially under the previous administration." But this is simply wrong; the word "exponentially" has a very specific mathematical meaning, one which we, now that we have studied exponentiation, know well.

When we looked at compound interest,[9] we saw that what makes it compound is that its increase depends on the value of the principle, which itself increases during every period. That is, its growth depends upon its current value.

exponential growth

growth which is dependent upon the current value of the thing growing; a prime example is compound interest

So now we know that the term "exponentially" is not just a synonym for "lots" or "very quickly"; it has a very specific meaning, which may or may not fit the situations to which it is applied.

6.2.2 Roots

Roots are the exact opposite of exponentiation; while we *raise* a base by a given exponent to acquire its power, we *extract* a *root* of a certain *degree* from the *radicand*.

The notation for roots is odd compared to the notation we know for other operations; but it becomes second nature before long. It centers on the symbol "$\sqrt{}$," called the *surd* or the *radical*. Any number underneath the radical is part of the radicand.

$$\left(b^x = y\right) = \left(\sqrt[x]{y} = b\right)$$

A more concrete example might be instructive:

[8] For more on perbiquas, see Section 6.3.4, at 126. [9] *See supra*, Section 6.2.1.4, at 100.

$$(2^3 = 8) = (\sqrt[3]{8} = 2)$$

So we begin by noting that the *answer* in an exponentiation problem (the *power*) is actually what we start with in a root problem. The term we use for this is the *radicand*:

radicand

the number from which a root is extracted; the x in $\sqrt[z]{x} = y$; equivalent to the *power* in an exponentiation problem

Then we note that the exponent is listed just to the left of the radical; in a root problem, this is called the *degree*:

degree

the exponent to which the root must be raised to equal the radicand; the z in $\sqrt[z]{x} = y$; if not explicitly noted, then 2

As noted in the definition, we often don't actually write the degree in a root problem; if it's not written, then it's equal to two. So $\sqrt{4}$ is the same as $\sqrt[2]{4}$. Furthermore, just as we noted that in exponentiation the power of a number raised to the exponent of 2 is called its *square*, the root of a number to a degree of 2 is called its *square root*.

Finally, the *base* of an exponentiation problem is equivalent to the *answer* of a root problem. The answer in a root problem is, unsurprisingly, called the *root*:

root

that number which, when multiplied by itself a given number of times, will produce a certain number; the y in $y = \sqrt[z]{x}$; the answer in a root problem

Note that there are no (real) roots of negative numbers to even-numbered degrees. This is because multiplying two negative numbers together gives a positive number, so *all* bases to the powers of multiples of two are positive, which means that no negative number can have an even-numbered root. Odd-numbered roots, however, are possible, because by multiplying the positive result of an even number of divisions by the negative base one last time yields a negative number again. To demonstrate, let's try it with 2:

$$-2^2 = -2 \cdot -2 = 4$$

Because the negatives cancel out. This means that:

$$\sqrt{-4} \neq -2 \cdot -2$$

No number multiplied by itself will *ever* equal -4, because the only way to get a negative result to a multiplication problem is to multiply a positive by a negative. It's impossible to get a negative when multiplying a number by itself. On the other hand,

$$\sqrt[3]{-8} = -2 \cdot -2 \cdot -2 = 4 \cdot -2 = -8$$

This one works perfectly well; the odd number of multiplications yields a negative result.

So to sum up: even degrees cannot be applied to negative radicands; in arithmetic, we say that these are *undefined*.[ʕ] However, odd degrees *can* be so applied.

Exercises 6.8

3Ɛ. You have learned that exponents and roots are two sides of the same coin. Write the names of the parts of each and say which corresponds to which. Answer

40. You have learned the root of a radicand to the degree of 2 is called its *square* root. What is the root of a radicand to the degree of 3 called? Answer

41. You have learned that "$2^3 = 8$" is the same as "$2 \cdot 2 \cdot 2 = 8$". Given that, what does "$\sqrt[3]{8} = 2$" mean? Answer

6.2.2.1 Roots as Fractional Exponents

Frequently in arithmetic we find that different operations can be expressed as other operations if we rephrase what we're doing a little bit. Roots are likewise subject to such rephrasing; roots are also *fractional exponents*.

That is, there is a literal and direct translation from a root problem to an exponentiation problem. The simplest is the square root:

$$\sqrt{b} = b^{\frac{1}{2}}$$

With a fractional exponent, the denominator of the exponent becomes the degree of the corresponding root problem, while the numerator of the exponent becomes an exponent on the radicand of the corresponding root problem. This can be made explicit in the simplest case (a square root) this way:

$$\sqrt[2]{b^1} = b^{\frac{1}{2}}$$

Or a more complex case:

$$\sqrt[4]{b^3} = b^{\frac{3}{4}}$$

[ʕ] There is a branch of arithmetic which makes such numbers defined, after a fashion; this is called *complex arithmetic* using *complex* or *imaginary numbers*. It is important in advanced mathematics; however, it is beyond the scope of this text.

If your exponent is a mixed number, you're faced with an unusual situation, but the solution is simple: turn the exponent into an improper fraction and the problem is again straightforward.

If the exponent is a digital fraction, you'll have to turn it into a vulgar fraction and solve accordingly. Remember that this is relatively straightforward:

$$b^{0;4} = b^{\frac{4}{10}} = \sqrt[10]{b^4}$$

Of course, if your digital fraction is nonterminating, you'll have to truncate or round it to make this operation possible.

So such transformation is possible; why bother to do it? Because the new forms are often easier to visualize and to solve than the originals. Take, for example, the problem $9^{0;6}$; how can one go about solving that? Can one multiply 9 by itself 0;6 times, which is the meaning of the problem when expressed as exponentiation? Of course not; it's impossible to do this directly. But when we remember that $0;6 = \frac{1}{2}$, and we rearrange the problem in the ways we've learned are possible, we can make it simple and easy:

$$9^{0;6} = 9^{\frac{1}{2}} = \sqrt[2]{9^1} = \sqrt{9} = 3$$

Juggling various ways of expressing a problem is often a way to make a seemingly intransigent problem into a solvable, or even an easy, one. Becoming facile with these methods is indispensable to becoming proficient with arithmetic.

Exercises 6.9

42. Rephrase the following; if expressed as a radical, rephrase as a fractional exponent, and vice verse. Answer

42(a) $\sqrt[3]{9}$ Answer **42(b)** $5^{\frac{6}{7}}$ Answer **42(c)** 4^4 Answer **42(d)** $\sqrt[2]{8^3}$ Answer

6.2.2.2 Extracting Square Roots

For the first time in our journey through arithmetic, we have come to a computation which is nothing but intelligent estimation and slogging through volumes of work. There is, unfortunately, no simple algorithm for extracting a root by hand; indeed, extracting a root of any degree other than two is nearly impossible. (By hand; digital computers, of course, have made these things easy.)

We find a square root by *trial and error*; that is, we make an intelligent estimation of the right value, try it, and then use the result to try again, gradually coming closer and closer to the correct answer. Once we've gotten "close enough" (as close as we need to given the nature of our problem), we stop.

Before dedicating ourselves to this task, though, it will be very helpful to memorize all the squares which are contained in two digits. A number whose square root is integral is called a *perfect square*:

perfect square

a number whose square root is a simple integer; that is, whose square root has no fractional parts

So let's look at those perfect squares and commit them to memory before we proceed:

1^2	=	1	2^2	=	4
3^2	=	9	4^2	=	14
5^2	=	21	6^2	=	30
7^2	=	41	8^2	=	54
9^2	=	69	↊^{2}	=	84
↋^{2}	=	↊1	10^2	=	100
11^2	=	121	12^2	=	144
13^2	=	169	14^2	=	194
15^2	=	201	16^2	=	230

Once these have been comitted to memory, we can proceed to try to extract an actual square root. Let's suppose we need to extract the square root of 35.

There are four steps to the problem:

1. Guess an approximate answer.
2. Divide the square by the guess.
3. Average the quotient from step 2 and the the guess.
4. Using result of step 3 as your new guess, repeat steps 2–4 until arriving at a "good enough" answer.

This requires us to define *average*:

average

the quotient of the sum of a list of addends and the number of those addends

In other words, to average a list of numbers, add them all up, then divide the sum by the number of numbers you added. In the case of extracting square roots, you will always be summing two addends, so you will always be summing them and dividing by two.

So we are considering 35. First, we take a guess; since 30 is a perfect square (its square root is 6), and 35 is greater than 30, we know that our square root will be greater than 6;

but since the square of 7 is 41, and 35 is less than 41, we know that our square root will be less than 7. So let's guess 6;6 and see where that gets us.

We divide 35 by 6;6:

$$
\begin{array}{r@{\,}l}
 & \,6;2\;1 \\ \cline{2-2}
6;6 & \big)\,3\;5;0\;0 \\
 & \,3\;3\;0 \\
 & 2\;0 \\
 & 2\;0\;0 \\
 & 1\;1\;0 \\
 & \text{Ɛ}\;0 \\
 & 6\;6
\end{array}
$$

And we'll stop there for now, though of course we could continue gathering digits. And that brings us to step 3: average this quotient (6;21) and our guess (6;6).

$$
\begin{array}{r}
6;6\;0 \\
+\,6;2\;1 \\ \hline
1\;0;8\;1
\end{array}
\qquad
\begin{array}{r@{\,}l}
 & 6;4\;0 \\ \cline{2-2}
2 & \big)\,1\;0;8\;1 \\
 & \,1\;0 \\
 & 0 \\
 & 0\;8 \\
 & 8 \\
 & 0
\end{array}
$$

The average, then, is 6;40, or 6;4. That's our new guess. So now we repeat, dividing the square by our new guess: 35 ÷ 6;4 = 6;58. Then we average our guess and that quotient: 6;58 + 6;4 = 10;97Ɛ; 10;97Ɛ ÷ 2 = 6;49. Our new guess, then, is 6;49.

Starting over: 35 ÷ 6;49 = 6;4Ɛ1; averaging that and our original guess: 6;4Ɛ1 + 6;49 = 10;981; 10;981 ÷ 2 = 6;4↊06. That's our new guess.

Starting over: 35 ÷ 6;4↊06 = 6;4↊08; averaging that and our original guess: (6;4↊08 + 6;4↊06) ÷ 2 = 6;4↊07. That becomes our new guess.

We can keep going as many times as we want, though of course, we need to use longer strings of digits each time. If we wanted to keep going, we'd get an increasingly accurate estimation of the true square root; to a dozen digits, the square root of 35 is

6;4↊07 2263 3879. But for almost all applications, four digits of precision is more than enough.

Exercises 6.↊

43. Find the square root of 25 to 3 digits. Answer
44. Find the square root of 13;4893 to 4 digits. Answer
45. Find the square root of 478↊3;↋↋ to 4 digits. Answer

6.2.2.3 Extracting Other Roots

So that's how we extract square roots, surely the most common type needed. But what about other roots; say, cube roots?

We again have to proceed by *iteration*: by taking an intelligent guess, trying it out, and then progressively getting closer to the true answer. Square roots are a special case, though, so we have to use a more complex method.

If we call our guess x, and the number from which we're trying to extract the cube root y, then we can calculate the result of our guess of the cube root with the following equation:

$$\frac{(2 \cdot x^3 + y)}{(3 \cdot x^2)}$$

Remember, a vulgar fraction is merely division by a different notation, so an equation written in the form of a vulgar fraction simply means division. So let's consider the number ↊84 and try to extract its cube root. We'll start with ↊ as our initial guess. Remember that each time we guess, we have the same value for y—↊84, the number in question—but a different value for x—whatever our current guess is. As said, we start with ↊. So we plug ↊ in for x and ↊84 for y, like so:

$$\frac{(2 \cdot ↊^3 + ↊84)}{(3 \cdot ↊^2)} = \frac{(11↊8 + ↊84)}{(210)} = \frac{2070}{210} = ↋;9724$$

Then we try another number close to our first answer and see what we get. Our series is as follows:

Try	Guess	Result
1	↊	↋;9724
2	↋	↋;6↊↊↊
3	↋;6	↋;66↋3
4	↋;7	↋;66↋1
5	↋;66	↋;66↊↋
6	↋;66↊↋	↋;66↊↋

As our result gets closer and closer to our guess, our answer is closer and closer to true. We could extend this answer to more uncial places, but there's no need here; we can be sure that this answer is quite accurate. And, indeed, checking by digital calculator confirms that we have the right answer. $\sqrt[3]{↊84} = ↋;66↊↋$.

We can use the same equation for other degrees of roots, simply by altering the exponents in it. The exponent in the numerator should be the same as the root we're seeking; the exponent in the denominator should be one less than the root we're seeking.

So let's look for the *sixth* root of ↊84, rather than the cube root. We'll start our guess with 4 this time.

$$\frac{(2 \cdot x^6 + y)}{(3 \cdot x^5)} = \frac{(2 \cdot 4^6 + ↊84)}{(3 \cdot 4^5)} = \frac{(48↊8 + ↊84)}{(1940)} = \frac{5770}{1940} = 3;2023$$

We tried 4, and ended up with 3;2023 as a result; so we must be off by a considerable margin. So let's try 5 now, and narrow down our result as much as we can:

Try	Guess	Result
1	4	3;2023
2	5	3;5↋7↊
3	3	4;1424
4	3;6	3;388↊
5	3;4	3;5776
6	3;46	3;5098
7	3;48	3;4↊86
8	3;49	3;4983
9	3;496	3;4923
↊	3;493	3;4953
↋	3;494	3;4943

And finally, we get to an answer to three digits that matches our result pretty well. And sure enough, checking the sixth root of ↊84 by calculator gives us a reasonably close answer: $\sqrt[6]{↊84} = 3;4941$.

Exercises on this would be more burdensome than helpful; it's good to know the method, but not necessary to practice. Extracting roots is a difficult, burdensome, error-prone process; the existence of digital calculators to ease this burden is a boon to arithmeticians everywhere.

6.2.3 Logarithms

Logarithms are an interesting branch of advanced mathematics with a close connection to exponents. Simply speaking, logarithms *are* exponents, but expressed in a different way, the same way that subtraction is really just addition of negative numbers. More formally:

logarithm

the inverse operation to exponentiation; the logarithm of a number is the exponent to which the base must be raised to produce that number

Logarithms are often called simply "*logs*," and that is the way we write them in equations: log. The notation is, again, quite different from that in other functions we've learned, but is easy enough to understand:

$$\log_b(x) = y$$

This is pronounced, "the logarithm y of x to base b," or "the log y of x to base b," or "the base-b log of x." y is the *logarithm*, b is the *base*, and x is the *power*.

Those students thinking back to our study of exponentiation[↋] will be struck by the similarity of terminology here: we are again talking of bases and powers, just as we did then. That is because, as mentioned earlier, logs are merely the inverse of exponentiation; the two are simply opposite ways of writing the same thing. This gives rise to a rule:

Equivalent Symbolism Rule

That logarithms are a different, but equivalent, symbology for exponentiation; i.e., that $b^x = y$ is equivalent to $x = \log_b y$.

We can use the Equivalent Symbolism Rule along with the Exponent Product Rule[10] and the Exponent Quotient Rule[11] to form similar rules regarding logarithms; namely, the Log of a Product Rule:

Log of a Product Rule

The log of a product is the sum of the logs of its factors; i.e., $\log(x \times y) = \log(x) + \log(y)$

And the Log of a Quotient Rule:

Log of a Quotient Rule

The log of a quotient is the difference of the log of its dividend and the log of its divisor; i.e., $\log(x \div y) = \log(x) - \log(y)$

Note that, when the base is not explicitly written, it is assumed to be 10. (In decimal, of course, it is assumed to be ↊; but in dozenal, we can assume a better number.)

[↋] *See supra*, Section 6.2.1, at ↋3. [10] *See supra* at ↋6. [11] *See supra* at ↋7.

All this is well enough; but how do we find logs? The answer is the same as when we studied extracting square roots[12]: trial and error. Over years, the mathematicians who discovered logarithms built up enormous tables of them, and the arithmetician would look up the values on these tables. These tables were used to make complex multiplication and division problems into much simpler addition or subtraction ones, which were then solved with the use of logarithmic tables. For example:

Let's consider a difficult multiplication problem: ↋45↊;3↊8 × 47;↋↋7. Solving this problem with paper and pencil is well within our abilities as arithmeticians at this point; but it's still a long problem, with lots of opportunity for careless errors.

But remember the Log of a Product Rule: the log of a product is the sum of the logs of its factors. So take the log of ↋45↊;3↊8, which is 3;↋8↊8↋2↋4; and the log of 47;↋↋7, which is 1;75322↋1↋; and add them, to make 5;721↋2213. That, then, is the log of the product of those two factors. We can find the actual product by taking that number's *antilog*:

antilog

the opposite of the log; that is, the number which, when made the exponent on the proper base, yields a certain power

This is, of course, another word for exponentiation. To calculate the antilog, simply raise the logarithm's base by the log; e.g., to calculate the antilog of 4;7823, calculate $10^{4;7823}$. The Equivalent Symbolism Rule makes this work.

Antilogs are traditionally, like logs, looked up on a table; so looking them up on the table, we see that the antilog of 5;721↋2213 is 45 0↊↋5;3269. (We can also do it manually, remembering that our base, when unspecified, is 10; so calculate $10^{5;721\text{↋}2213}$ to get the anwer.) If we go through the trouble to multiply ↋45↊;3↊8 by 47;↋↋7, we'll see that it equals 45 0↊↋5;3404, which is awfully close to our log-and-antilog answer. We could have kept listing digits of the logs as long as we wanted and gotten increasingly close to the right answer; in this way, we get "close enough" with a minimal amount of work.

Let's track this process more closely:

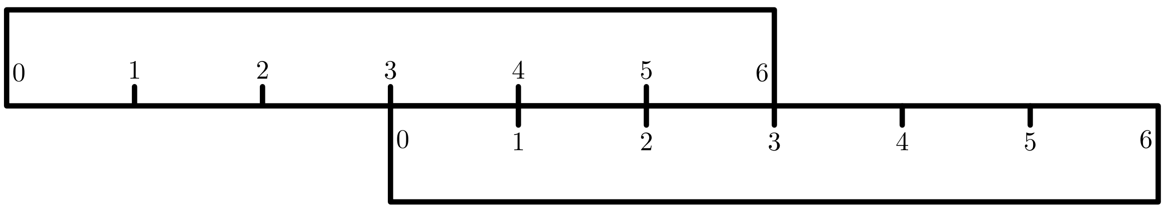
$$\text{↋45↊;3↊8} \times \text{47;↋↋7} = ???$$
$$\log(x) + \log(y) = \log(x \times y)$$
$$\log(\text{↋45↊;3↊8}) + \log(\text{47;↋↋7}) = \log(x \times y)$$
$$\text{3;↋8↊8↋2↋4} + \text{1;75322↋1↋} = \log(x \times y)$$
$$\text{5;721↋2213} = \log(x \times y)$$

So now we know what the log of the product is. We can then take that log's antilog, which gives us 45 0↊↋5;3269. The true answer, multiplied directly, is 45 0↊↋5;3404; not exactly right, but close, and we could get it closer by using more digits in our logarithms.

[12] *See supra*, Section 6.2.2.2, at 105.

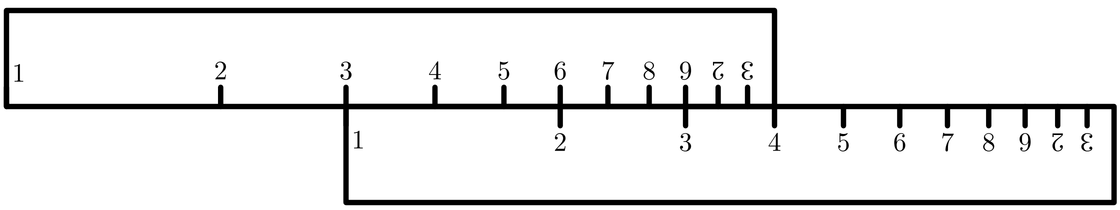
For generations, mathematicians used logarithms in precisely this way: as a way of converting complex and difficult multiplication and division problems into simpler addition and subtraction ones. Huge tables of logarithms were developed to assist this endeavor.

The *slide rule* is a device which takes advantage of this property of logarithms. By running two rulers against one another, each of which is marked on a *linear* scale, we can easily do addition and subtraction without manual calculation:



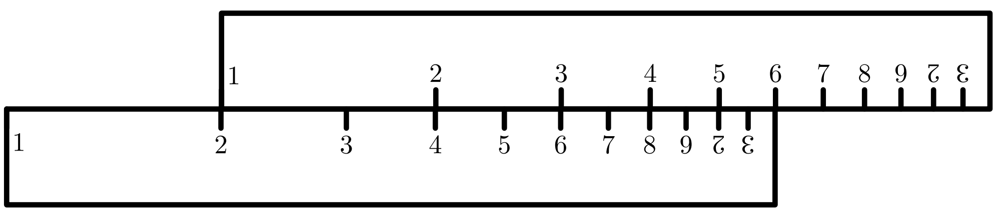
Note that once we have the 0 on the lower ruler aligned with the three on the upper ruler, the numbers on the lower ruler align with their sums (with 3) on the upper ruler. In other words, when 0 and 3 are aligned, then the lower ruler's 1 points at the sum of 3 and 1 (4); the lower ruler's 2 points at the sum of 3 and 2 (5); and so on.

But the slide rule goes one better, utilizing *logarithmic* scales. By spacing the marks on the rulers logarithmically rather than linearly, the slide rule enables multiplication and division, though the user must know where to put the dit.



Here, when the 1 on the lower ruler is aligned with the 3 on the upper ruler, the lower ruler's number is aligned with the *product* on the upper ruler. That is, with the 1 and the 3 aligned, the lower ruler's 2 aligns with the product of 3 and 2 (6); the lower ruler's 3 aligns with the product of 3 and 3 (9); and so on.

To go higher than this, some tricks are required, which we'll demonstrate on the slide rule below. Say you wish to multiply 6 by 6. Align the upper ruler's 6 with the end of the lower ruler, for example, and you will see the products of 6 line up underneath their factors on the upper ruler. Notice that the 2 on the lower lines up with the 1 on the upper. You need to place your own dit; that requires you to add a zero to 1, giving you 10. So you know that $6 \times 2 = 10$. Now find the 8 on the lower ruler, and see that it is aligned with the 4 on the upper ruler. Place your own dit; $6 \times 8 = 40$.



(These slide rules are horrifically simplified. True slide rules are precision instruments, beautifully precise, capable of calculating almost anything, and sufficiently accurate to take American astronauts to the Moon and back. Here, we're just demonstrating the concept.)

So simply by switching from a *linear* to a *logarithmic* scale, we've switched from addition to multiplication; and that is the chief benefit of logarithms. Logarithms were used for generations to make these routine calculations much easier for people to do.

Exercises 6.Ɛ

46. Using the Log of a Product Rule and Log of a Quotient Rule, calculate the following. Use of a digital calculator makes sense to determine the logs; the arithmetic itself should be done by hand. Four digits of the log is sufficient. Answer

46(a) 87;389 × 944;3Ɛ3 Answer **46(b)** 4;3 × 3;Ƨ Answer **46(c)** 3;2 ÷ 6;8 Answer
46(d) 23;876 × 86;4 Answer **46(e)** 187;3 ÷ 4;1 Answer **46(f)** 14 ÷ 4 Answer

6.2.3.1 Natural Logarithms and e

So far, we've discussed only *base-10* logarithms; that is, logs with a base of 10. But just as we can have exponents of any number, so we can have logarithmic bases of any number. The base is written as a subscript after the word "log"; e.g., $\log_3(5)$ references the logarithm of 5 to base 3.

There is one special base, though, so unique that it is given a special symbol: e. This is called "Euler's number," and is described as the base of the *natural* logarithm:

Euler's number

a nonterminating fraction which is the limit of $(1 + \frac{1}{n})^n$ as n approaches infinity; equal to 2;8752 3606 9821 . . . ; symbol e

Limits are a concept of precalculus and are well beyond our scope here; the important thing to know about e for our purposes is that it is the base of the *natural logarithm*:

natural logarithm

logarithms to the base of Euler's number; symbol ln

To try to understand this, think back to our discussion of compound interest.[13] While we discuss this, remember the Equivalent Symbolism Rule; when we're talking about logarithms, we're just talking about exponentiation in a different way, so these fields will always be closely related.

We saw that we can use exponentiation to calculate compound interest; in our example, that interest accrued monthly. However, what if our interest rate was 1, and we wanted it to accrue weekly? Daily? Hourly? As the intervals between the accrual get smaller and

[13] *See supra*, Section 6.2.1.4, at 100.

smaller, the closer the rate of change approaches to this value of e. For example, if you begin with \$1, and compound monthly (10 times per year), you will end with \$2;7638. If you compound weekly (44 times per year), you will end with \$2;8569. If you compound twice weekly (88 times per year), you will end with \$2;8569. If you compound daily (265 times per year), you will end with \$2;86↊9. Twice daily, \$2;871↋. Notice that the more often we accrue, the closer the ending value gets to 2;8752 3606 . . .

If we could somehow compound *continously*, all the time without stopping, we would arive at precisely the value of e as our final value. Since it is continuous, it applies to a lot of natural phenomena that don't happen in neat, orderly time frames, like population growth, rate of decay, heating and cooling, and so forth.

e arises constantly throughout nature; it is extremely important in probability theory; population growth; interest; statistics. It has interesting properties in calculus. All told, its presence is so all-pervasive, and it arises naturally in so many circumstances, that logs to it as a base are called *natural logarithms*.

Natural logarithms are so important that we write them with a different notation than we do logs of any other base:

$$\log_e(x) = \ln(x) = \ln x$$

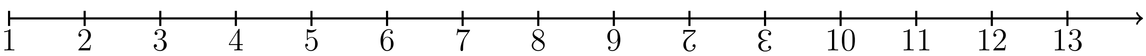
This means that $\ln e = 1$, because $\ln e = 1$ means $\log_e(e) = 1$, which itself means $e^1 = e$; and we know, thanks to the Rule of Exponential Identity, that any number raised to the first power is itself.

Further explorations of natural logarithms is beyond the scope of this little book. But know that they are a rich field of study, both for abstract mathematics and for engineering and practical sciences.

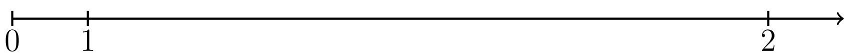
6.2.3.2 Logarithmic Scales

We've already seen that it's frequently helpful to deal in logarithmic scales rather than linear ones; the specific example we saw was the slide rule. Many such logarithmic scales are based on base-10 logarithms (or, in the decimal system, base-↊ logarithms). A scale is *logarithmic* when each mark on the scale follows a logarithmic progression; it is *linear* when it follows a mere progression of digits.

Consider the following number line:



Note it proceeds in an orderly manner, with the space between each number precisely the same as the space between any other number. This is a *linear* scale. Now consider the following:



Note that this does *not* proceed with the space between each number the same as that between any other number. In fact, the higher we go, the more space between numbers there are. Each whole number is an *order of magnitude* larger than the last, such that, if we were to put 3 on it, it would be well off the right side of the page. When we say an *order of magnitude*, we mean (more or less) adding a 0 to the right of the number. In other words, 0 is 0, 1 is 10, 2 is 100, and so forth.

A commonly-used and at least somewhat understood example of such a scale is the Richter scale, used to measure the strength of earthquakes. This is a decimal logarithmic scale, so an earthquake measured as 5 on the Richter scale is ↊ times stronger than one measured as 4. This type of scale enables us to fit a very large variance of possible values into a much more compact space.

It is *logarithmic* because it is *exponential*; that is, the exponent becomes the value we cite. Consider the following:

0	10^0	1
1	10^1	10
2	10^2	100
3	10^3	1000
4	10^4	1 0000
5	10^5	10 0000
6	10^6	100 0000

Quite simply, we use the *exponent* rather than the raw number. To illustrate this, let's contemplate a completely fabricated scale of measurement for noise, where 1 is the softest noise the human ear can detect. We could count increases of volume individually; or we can count them logarithmically. So we turn that softest noise the human ear can detect into a unit; let's call it the *blip*. Now, standing next to a jet engine might be many triqua of blips; let's say, for the sake of argument, that it's 1000 blips. We can then say that the noise is 3 *logblips*, using the exponent to create a logarithmic unit to better handle the scale.

Let's say we've measured a certain sound and found it to be 3;4 logblips. To bring this measurement into normal blips, we must remember that the logblips are just the *exponents* of the blips:

$$10^{3;4} = 2358;1709$$

So we have (roughly) 2358;1709 blips. We can tell that this answer is likely correct; 3 would give us 1000, and 4 would give us 10000, so this figure seems about right.

If, on the other hand, we know the number of blips but we want to find out what that quantity is in logblips, we do the opposite:

$$\log 2358 = 3;3\text{↋↋}$$

We've rounded here, so the answers aren't identical; but we can clearly see that these two units, blips and logblips, convert cleanly into one another.

Logarithmic scales help us manage measurements in which there is a very large range of possible values; no matter how large or small our measurement is, we can keep it within reasonable, easily-handled ranges.

Logarithmic scales also give us an immediate idea of order of magnitude without having to count digits. When we see a noise of 4;7 logblips, we know that it is two orders of magnitude louder than a noise of 2;7 logblips; that is, that it's 100 times louder.

6.2.3.3 Base-2 Logarithms

In addition to our normal base-10 logs and the special natural logarithms, there remains one type of logarithm that we should study individually: base-2 logarithms, or *dublogs*.

dublog
logarithms to base 2

We should study dublogs not only as a special case of logarithms in general, though they are interesting in that sense, but also as the basis of systems of measurement in some dozenal metric proposals, including TGM.[14]

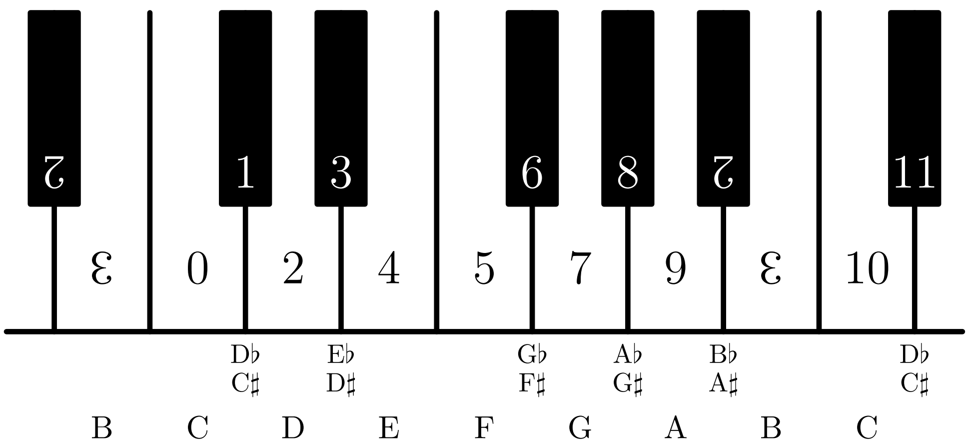
Quite commonly, daily tasks require *doubling* and *halving*, and these operations are the most common performed in certain fields. In such fields, even when not actually doubling and halving, the operations actually depend on the size of a doubling or a halving. The most obvious case is musical frequencies.

In music, we divide notes into *octaves*; but despite the name, the number 8 has very little to do with it. An octave is, in fact, a series of *twelve* (10) notes; and the most important thing to note is that the twelfth note is, in fact, *double* the frequency of the first, while the sixth note's freqency is the first's times the square root of two.

If we take the twelfth root of 2 ($\sqrt[10]{2}$, $2^{\frac{1}{10}}$, or $2^{0;1}$), we get 1;0869 (when rounded to four digits). The base-two logarithm (dublog) of 1;0869 is . . . 0;1.

Indeed, the dublog of *every* multiple of the twelfth root of 2 equals a simple uncial fraction. That is, the dublog of $(\sqrt[10]{2})^2 = 0;2$, the dublog of $(\sqrt[10]{2})^3 = 0;3$, and so on, all the way up to the dublog of $(\sqrt[10]{2})^{\text{↋}} = 0;\text{↋}$. That much is interesting but unremarkable; but what *is* remarkable is that these powers of the twelfth root of 2 align quite closely to the semitones of the octave.

Consider the following set of piano keys:



Let's call the note of middle C zero (0). Counting from 0, we see that the next G is 7, so let's refer to that note as 0;7; it's frequency is not 0;7 times that of note 0, but it's frequency *is* 0;7 *dublogs* of that of note 0; so by raising 2 to the power of 0;7 ($2^{0;7}$), we

[14] *See* TGM: A Coherent Dozenal Metrology 60–6↊, available at http://www.dozenal.org.

can determine what relationship between note 0 and note 0;7 really is (note 0;7 is 1;5Ɛ91 times the frequency of note 0, which we calculate by noting that $2^{0;7} = 1;5Ɛ91$; and we reverse that calculation by saying that $\log_2(1;5Ɛ91) = 0;7$).

We can say, then, that G has a frequency 0;7 *Doubles* that of C, since every increase of 1;0 in dublogs is a doubling.

So let's check our answer. Drawing from canonical frequency figures, middle C is 199;7618 hertz, while the next G up is 287;ƐƐ34 hertz. 287;ƐƐ34 ÷ 199;7618 = 1;5Ɛ90, the dublog of which ($\log_2(1;5Ɛ91)$) is 0;7. It all checks out.

Why bother? Why not just use the frequencies of the notes themselves? For the same reason we ever bother using logarithms: they allow us to replace complex, long, error-prone calculations with shorter, simpler ones.

Imagine working directly with frequencies. We've already seen that the frequency of middle C is 199;7618 hertz. Tell me the frequency two Cs above it. We know that each C is a *doubling* of frequency of the last one; so all we really need to do is 199;7618 × 2 × 2, yielding the answer 732;6068 readily. With dublogs, though, you can give me the frequency of middle C—let's say, as before, 0—and I can tell you that two Cs up is 2. Each increase of a whole number is a *doubling* of frequency. I can then calculate the actual difference in frequency later, if I need to (usually I won't), by noting that $2^2 = 4$, so the difference in frequency is 4 times. (Check: 199;7618 × 4 = 732;6068, right on target.)

There is also the matter that *half* a doubling does not mean the same thing as half-again the original number; that is, C's frequency may be 100, and the next C's 1000, but halfway between the two is not 600. Dublogs enable us to talk about these intervals much more easily.

For example, we know that middle C is 199;7618, and that the next C up from it is 377;3018. But that doesn't mean that F♯, the note precisely midway between them, has a frequency exactly midway between their frequencies. If it did, its frequency would be (377;3018 − 199;7618)/2 + 199;7618, or 288;5318; but the actual frequency is 269;ƐƐ17. Because the relationship between these individual points is logarithmic, not linear, simple linear arithmetic can't give us the true values.

However, if we remember that F♯ is 0;6 of a Double above middle C, we can easily calculate the frequency as $2^{0;6} = 1;4Ɛ79$; 199;7618 × 1;4Ɛ79 = 269;ƐƐ05, which is slightly off due to our rounding, but clearly shows that we're on the right track. And it is much easier to say, "increase the frequency by half a Double," or by 0;6 of a Double, than to worry about what the precise frequency value is.

Another commonly considered example is paper sizes. The SI metric system uses the A-series and the B-series, but for now, let's concentrate on a dozenal system of measure, TGM, and the paper series it establishes, the Grafut-series (abbreviated "Gf") and the Surf-series (abbreviated "Sf"). Moving up and down both of these series involves doubling or halving the size of the paper, and so is an excellent demonstration of Doubles and dublogs.

The Grafut-series starts with the Gf+0, which is a sheet of paper one Grafut long and 1;5 Grafut wide. (This is almost identical to a metric A4 sheet.) The table on page 116 shows part of the Grafut-series in terms of its doubling and halving.

This shows that, even though the *ratios* of the sizes are easily understood, the actual sizes themselves are not. It is much more convenient to say "2;4 Doubles" than it is to say "having an area of 5;0586 times the original."

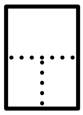
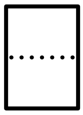
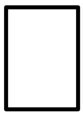
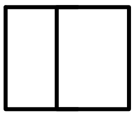
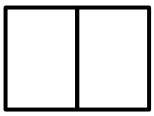
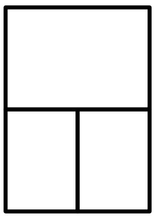
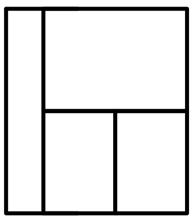
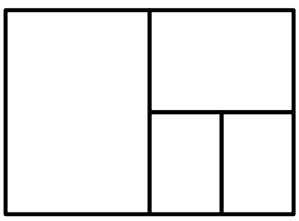
Paper Sizes in Doubles and Halves				
	Ratio	Name	Doubles	Area
	2 halves	Gf-2	-2	0;3
	1 half	Gf-1	-1	0;6
	0	Gf+0	0	1
	0;6 double		0;6	1;4Ɛ79
	1 double	Gf+1	1	2
	2 doubles	Gf+2	2	4
	2;4 doubles		2;4	5;0586
	3 doubles	Gf+3	3	8

Table 6.3: A demonstration of dublogs and paper sizes.

6.3 Ratio and Proportion

In our discussion of logarithms and especially of dublogs, we discussed that it was easier to describe the *ratio* in such circumstances than it was to describe the actual size. We offered no definition of "ratio" then, allowing context to fill in an intuitive understanding; now we will enter a formal exploration of ratios, and discuss how we can use them in daily life.

6.3.1 Ratios

To begin with, of course, we must define what a ratio is. A ratio is simply the relationship of one quantity to another:

ratio

an expression of the relationship of one quantity to another

Simple enough, of course. When we describe a ratio, we typically use a colon, ":", as our symbol (though there are other ways, which we'll get to in a moment). Consider the following ratio:

$$3 : 4$$

The first term of the ratio, in red, is called the *antecedent*; the second term, in blue, is called the *consequent*:

antecedent

the first term in a ratio

consequent

the second term in a ratio

We can describe the relationship between the antecedent and the consequent in English as, "3 to 4"; or, more specifically, in terms of the actual quantities involved. For example, $3 : 4$ might describe the ratio of salt to pepper in some particularly odd and spicy dish. In that case, we might say, "three parts salt to four parts pepper," or even replace the word "parts" with the actual unit involved, as in "three teaspoons salt to four teaspoons pepper."

Ratios can also be described as *vulgar fractions* or *digital fractions*. For example, our $3 : 4$ ratio can be expressed in the following ways:

$$3 : 4 = \frac{3}{4} = 0;9$$

Remembering that vulgar fractions are just division, we can construct the digital fraction expression of a ratio simply by dividing the antecedent by the consequent; the quotient is the ratio.

When a ratio is expressed as a digital fraction, it is customarily pronounced as such. The ratio above, 3 : 4, when expressed as a digital fraction, would be expressed as "the ratio of salt to pepper is 0;9," or even simply "0;9 salt to pepper." We know what digital fractions are, of course, so we know that this expression means "the ratio of salt to pepper is $\frac{9}{10}$," which equals $\frac{3}{4}$, and our notations are thus all brought together.

Finally, some ratios are special, and describe the relationship of two different types of quantities. These ratios are call *rates*:

rate

a ratio describing the relationship of two different types of quantity; e.g., miles to hours

Speeds (e.g., miles to hours, meters to seconds) are probably the most frequently encountered rates in the modern world, but this type of ratio is extremely common.

Ratios may relate more than two types of quantity, in which case they are listed separated by colons: 3 : 2 : 1.

Also, note that many ratios may be expressed backwards or forwards; e.g., our 3 : 4 ratio of salt to pepper could just as easily be described as a 4 : 3 ratio of pepper to salt. It's important, in such cases, to be clear which substance or quantity is the antecedent and which the consequent.

Exercises 6.10

47. Express the following relationships are ratios, using the colon format, and tell whether they are simple ratios or also rates. Answer

- **47(a)** An air rifle shoots a projectile which travels 130 feet for every second it moves. Answer
- **47(b)** A concrete mixer knows that the best concrete is produced when he combines one part cement, two parts sand, and four parts gravel. Answer
- **47(c)** A chemist knows that each water molecule consists of two atoms of hydrogen and one of oxygen. Answer

6.3.2 Calculating with Ratios

Oftentimes, we are presented with an appropriate ratio and asked to relate it to a specific quantity. For example, we can return to our example of a recipe consisting of three parts salt and four parts pepper. The ratio tells us how much salt compared to how much pepper; but how much actual salt and pepper are we putting into the recipe?

Suppose this is a medium-sized portion of soup that we're making, which requires a tablespoon of salt. We've been told that we should put in four parts pepper to three parts salt, but that's not part of the written recipe; it's the secret part that our mothers taught us. How can we tell?

By setting up the ratio as a vulgar fraction and comparing it to another vulgar fraction. Our 4 : 3 ratio of pepper to salt is expressed as follows:

$$\frac{4}{3}$$

The antecedent (4) refers to the pepper, while the consequent (3) refers to the salt. We know that the amount of salt we need is one tablespoon. So we can set up another ratio consisting of the actual amounts of salt and pepper, which must be equal to the vulgar fraction of our current ratio.

$$\frac{4}{3} = \frac{x}{1}$$

Here, we're using x to refer to the *unknown quantity*; namely, the amount of pepper in the recipe. We often use letters to represent an unknown quantity, or *variable*, in equations; what you're seeing here is really algebra, and anything beyond its most basic operations is beyond our text. For this, however, it will be essential.

So in red we have the quantity of pepper, and in blue we have the quantity of salt in tablespoons. Treating the ratios simply as vulgar fractions, how can we determine the value of x?

By *cross-multiplying*. That is, multiply the numerator of the first vulgar fraction by the denominator of the second, and vice versa:

$$\frac{4}{3} = \frac{x}{1}$$

$$4 \times 1 = x \times 3$$

$$4 = 3x$$

$$3x = 4$$

Once we get here, we seem to have a conundrum; $3x = 4$ seems impossible to solve. But remember that here we have an equation; and an equation retains the same value as long as we do the same operations to both sides of the equals sign.

Remember that any number divided by itself is 1, and any number multiplied by itself is itself; so we can get rid of the three by dividing $3x$ by 3. $3 \div 3 = 1$, so we get $1x$; and since $1 \times x = x$, we can express this as simply x. Then, because we must do the exact same thing to both sides of the equation, we divide 4 by 3:

$$3x = 4$$

$$\frac{3x}{3} = \frac{4}{3}$$

$$x = \frac{4}{3}$$

$$x = 1;4$$

This gives us our final pair of ratios:

$$\frac{4}{3} = \frac{1;4}{1}$$

In other words, if our ratio of pepper to salt is $4 : 3$, and the amount of salt we put in is one tablespoon, then we must put in 1;4, or one and a third, tablespoons of pepper.

In this case, we were fortunate to know the proper quantity of one of the substances; what if we only know the *total* amount, and neither individual amount? For example, what if we know that there are 100 sandwiches, and 100 people divided into two rooms, with $3 : 5$ people in each room?

We determine the total number of parts, then solve for each individual part. In our sandwich example, we know that there are eight parts ($3 : 5$; $3 + 5 = 8$). We also know that our total number of people is 100. So we determine the size of each part:

$$\frac{100}{8} = 16$$

So we have eight parts of 16 people each. Since we know that the ratio of $3 : 5$ applies here, we can find the number of sandwiches which need to go to each room simply by multiplying each value in the ratio by the size of each part:

$$\frac{3}{5} \times \frac{16}{16} = \frac{46}{76}$$

(Note that this does *not* change our value; $\frac{16}{16} = 1$, so we're really just multiplying by 1, which doesn't change the value. It just expresses the same value in a different way.)

So 46 sandwiches need to go to the smaller room, and 76 to the larger room. We can check our answer by seeing if the two ratios are equal by cross-multiplying again:

$$\frac{3}{5} = \frac{46}{76}$$

$$3 \times 76 = 5 \times 46$$

$$1↊6 = 1↊6$$

And we can see that it all works out. Now we can calculate exact amounts when we have the ratio and either one of the two amounts, or the total combined amount. This will enable us to solve many interesting arithmetical problems.

Exercises 6.11

48. If you added half of Jim's age to his actual age, you'd get 30. How old is Jim? Answer
49. If three pounds of chocolate is \$3;60, how much is 4 pounds? Answer
4↊. What number which, having been increased by $\frac{3}{7}$, will equal 76? Answer
4↋. You are the administrator of an estate, and the decedent has two heirs whom he wishes to inherit in a ratio of 2 : 5. The total estate is worth \$4↊54. How much does each individual receive? Answer
50. You are a general partner in a business, and your arrangement is that your partner, who invested $\frac{2}{3}$ of the business's assets, gets $\frac{2}{3}$ of the profits. The business's profits for the day are \$↊3970. How much of that profit is your share? Answer
51. A 20-foot beam needs to be divided into two parts with a ratio of 2 : 3. How long is each part? Answer
52. In September, 16 days were cloudy. What is the ratio of cloudy days to sunny days? Answer

6.3.3 Proportions

Proportion is another type of relationship between two numbers. However, *proportions* are tied to the rates of change of two variables in relationship to each other, while ratios simply express the relation with no reference to change.

> ***proportion***
>
> the relationship of two quantities such that a change in one always induces a change in the other, when such changes are related by a constant multiplier

This is a very wordy definition for expressing a very simple thing. Essentially, a proportion is a ratio consisting of two values in a specific relationship. In that specific relationship, changing one of the values will always change the other. Further, the changes in the two values are always related by the use of some constant value.

Consider, for example, an automobile and gasoline. When a driver hits the gas, the car speeds up. An increase in gas delivered to the engine *always* results in an increase in the engine's speed; it's impossible to increase the speed without increasing the gasoline headed to the engine, and it's impossible to increase the gasoline headed to the engine without increasing speed. Furthermore, we can reliably predict how much gasoline is needed for a certain increase in speed, and conversely how much of an increase of speed will be produced by a given amount of gasoline.[15] This means that gasoline and speed are in proportion to one another; we also say that they are *proportional* to one another.

[15] This is much simplified; we can increase the car's speed by going downhill, for example, without hitting the gas. But the idea is still contained here.

To say that two numbers are proportional to one another, we write:

$$x \propto y$$

There are two types of proportion, direct proportion and inverse proportion. We will address each of these in turn. However, first we will address a more basic form of proportion, which doesn't really relate to the definition we saw above, but which is nevertheless very useful for dealing with proportions later on.

6.3.3.1 Proportions as Equalized Ratios

Older mathematical texts will also define proportions as two ratios set equal to one another. E.g., the following would be considered a proportion, even though it may not qualify as such under the definitions of direct proportion or inverse proportion which we'll see shortly[16]:

$$\frac{4}{6} = \frac{14}{20}$$

The first and last terms of such a proportion are called the extremes; the second and third are called the means.

An interesting property of such proportions is the the Proportion Product Rule:

Proportion Product Rule

in a proportion understood as two ratios set equal to one another, the product of the means equals the product of the extremes

This means that, in our proportion above, the product of the two red numbers equals the product of the two blue numbers:

$$\frac{4}{6} = \frac{14}{20}$$

$$4 \times 20 = 14 \times 6$$

$$80 = 80$$

An informal name for this process is *cross-multiplying*, for obvious reasons: we are multiplying the numbers across the equals sign. And now we understand why cross-multiplying, which we first met in Section 6.3.2, actually works; and we have a more formal name, the Proportion Product Rule, for the concept and process. Now, we can proceed to meet the more formal types of proportion: ratios related specifically to change.

Note that this property only works if the two ratios are actually equal; if they're not, this won't work.

[16] *See infra*, Section 6.3.3.2, at 121, and Section 6.3.3.3, at 124.

6.3.3.2 Direct Proportion

Two values are said to be *directly proportional* if there is some number, not equal to 0, such that one of the values multiplied by that number is always equal to the other number. That is, we can say that x is directly proportional to y is there is a third number, k, for which $y = k \cdot x$.

direct proportion

a proportion in which one variable is always the product of the other and a constant; y and x are directly proportional if $\frac{y}{x}$ is constant

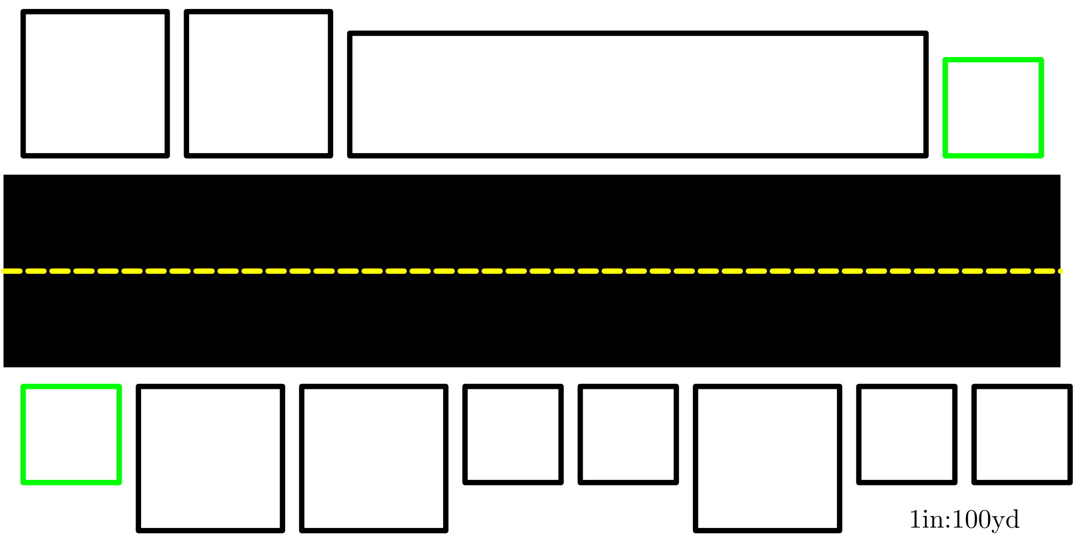
Even more *algebraically*, we can define it thus: given two values x and y, y is directly proportional to x if there is a non-zero constant k such that $y = kx$.

k can be any value at all; but it is *constant*, such that multiplying one value of the proportion by k will always equal the other value of the proportion. That value is called the *proportionality constant*:

proportionality constant

the non-zero constant which links two directly proportional values; also known as the "constant of proportionality" and the "constant of variation"

Let's consider for a moment a map drawn "to scale"; that is, where the distances on the map directly reflect the real distances between objects. For example, here is a map depicting a fictional street in a fictional city:



Note, in the lower-left corner, the phrase "1in:100yd". That is referred to as the *scale*: the first number indicates the distance on the map, and the second indicates the distance in the real place. In other words, every inch on this map corresponds to 100 yards of real distance on the real street. (Or would, if the street were real.) It's a *ratio*, of course, a ratio of 1 : 100, or $\frac{1}{100}$; but more importantly for our purposes it's the *proportionality constant*.

What this means is that the distance between any two points on the map (say, the green buildings in either corner) is directly proportional to the distance between the two real locations in reality, with the scale being the proportionality constant.

The distance between the two green buildings on the map is five inches (5 in). This distance is directly proportional to the real distance between the two buildings on the actual street. Since our proportionality constant lists the map distance followed by the real distance, we'll keep it that way. The map distance is y, and the real distance is x.

So direct proportionality means that $y = kx$. Let's plug those values in:

$$y = kx$$

$$5 = \left(\frac{1}{100}\right)x$$

$$5 = \left(\frac{1}{100}\right)\left(\frac{x}{1}\right)$$

Juxtaposition is just multiplication; so let's multiply.

$$5 = \frac{x}{100}$$

We need to get the x by itself, which means we need to get rid of the denominator 100 here. Remembering that fractions are just division, and that the opposite of division is multiplication, and that as long as we do the same thing to both sides of the equation, we can get rid of the denominator by multiplying both sides of the equation by it; that is, by 100.

$$5 \times 100 = \frac{x}{100} \times 100$$

$$500 = \frac{x}{100} \times \frac{100}{100}$$

$$500 = \frac{100x}{100}$$

$$500 = x$$

That's 500 *yards*, not inches; the proportionality constant, containing the units in it, took care of that for us.

Conversely, if we know that the real distance is 500 yards and want to know the distance on the map:

$$y = kx$$

$$y = \frac{1}{100} \times 500$$

$$y = \frac{1}{100} \times \frac{500}{1}$$

$$y = \frac{500}{100}$$

$$y = 5$$

We need only know our proportionality constant, and knowing one value means that we know the other; and one value is always the product of the proportionality constant and the other value.

Direct proportion is encountered frequently in science and engineering; for example, the gravitational attraction between two objects is directly proportional to the product of their masses, and the electrical attraction (or repulsion) of two particles is directly proportional to the product of their charges.

Exercises 6.12

53. Tell whether the following are directly proportional; and if so, what the proportionality constant is. Answer

- **53(a)** An airplane is travelling at 400 miles per hour. Some of the pilot's instruments have malfunctioned, and he is interested in determining how far he has travelled and how long he has taken to travel it. Answer
- **53(b)** An astronaut on the International Space Station drops his drill while on a spacewalk. He knows the mass of the drill and the acceleration of gravity at his altitude; he wants to determine how much force is pulling the drill towards the earth. The equation to determine this is acceleration equals force divided by mass. Answer
- **53(c)** A business manager is trying to determine the appropriate price for his company's new product. It's a new type of product, so he's not sure what people are willing to pay for it, or even what the demand will be. He thinks that manufacturing costs should be an important factor in setting a price, as well as what the product's predecessor (a different type of product) had sold for. Answer
- **53(d)** A young geometry student knows that the ratio between the diameter of a circle and its circumference is constant; increasing the diameter means an increased circumference, and vice versa. He knows that he can get the circumference by multiplying the diameter by π. Answer

54. π is (approximately) equal to 3;1848, and it is the proportionality constant between the circumference of a circle and its diameter; the diameter of the circle is directly

proportional to its circumference. Round your answers to four digits, if necessary.
Answer

54(a) If the diameter of the circle is 43, what is its circumference? Answer
54(b) If the diameter is 32;32, what is the circumference? Answer
54(c) If the circumference is Ɛ2, what is the diameter? Answer
54(d) If the circumference is 13, what is the diameter? Answer

6.3.3.3 Inverse Proportion

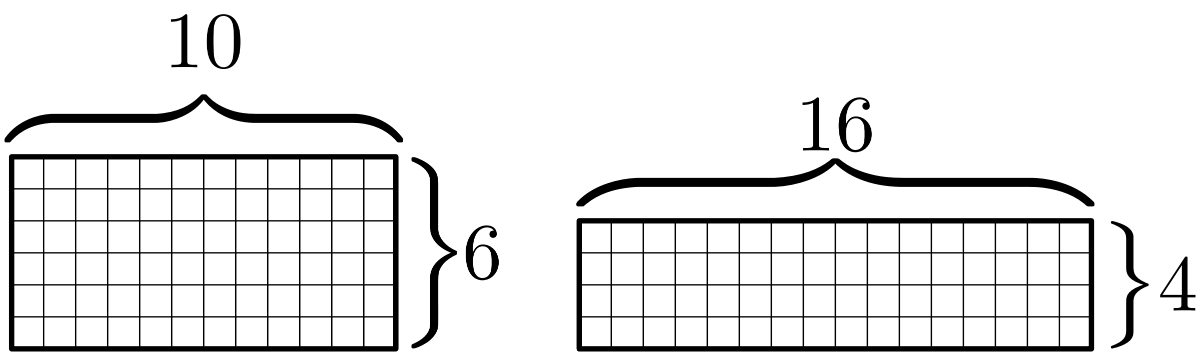
Inverse proportion differs from direct proportion in that, rather than the two values being linked by a proportionality constant, they are linked by their product, which is therefore itself the proportionality constant.

Two values are said to be inversely proportional if their product is always the same value. This means that if one of the values increases, the other decreases such that the product of the two will still be the same. As a result, the term "proportionality constant" has a different meaning for inverse proportion than for direct proportion:

> ***proportionality constant***
>
> for inverse proportion, the product of the two values which are inversely proportional to one another; this product is constant regardless of the changes in the values themselves

So in inverse proportion, when one of the values increases, the other decreases, and vice versa. An excellent example is the relationship of the length and width of a rectangle, assuming a constant area. Remember that the area of a rectangle is the product of its length and its width.

For example, let's assume we have a rectangle which must retain an area (the product of the length and the width) of 60 units:



Note that both 10×6 and 16×4 both equal 60. Here, the length and width of the rectangles are inversely proportional to one another, because their product always remains the same; the area of the rectangle is the proportionality constant. (This isn't true of *all* rectangles; just these hypothetical ones, where increasing the length decreases the width so as to preserve its area.)

Inverse proportion gets its name from the fact that it's literally the inverse of direct proportion; that is, saying that y is *inversely proportional* to x is also saying that y

is *directly proportional* to $\frac{1}{x}$. We can prove this with equations; x and y are directly proportional if $y = kx$ (that is, they are related by a proportionality constant), and they are inversely proportional if $yx = k$ (that is, their product is the proportionality constant). Using algebra (again, beyond the scope of this text), we can make y the product in both equations:

$$yx = k$$

$$\frac{yx}{x} = \frac{k}{x}$$

$$y = \frac{k}{x}$$

$$y = \frac{1}{x}k$$

By dividing both sides of the equation by x, and then noting that $\frac{1}{x}k$ is equal to $\frac{k}{x}$ (multiply the first if you don't believe me), we produced the final equation.

But note that $y = \frac{1}{x}k$ looks a lot like $y = kx$, except that x is replaced by $\frac{1}{x}$. This is the *reciprocal*, or inverse, of x; hence, they are *inversely proportional*.

Inverse proportion is quite common in science and engineering, particularly the Inverse Square Law, in which one quantity is inversely proportional to the square of another. For example, the gravitational attraction between two objects is inversely proportional to the square of the distance between them; and the attraction (or repulsion) between two electrically charged particles is inversely proportional to the square of the distance between them.

Exercises 6.13

55. You are painting your back fence. You know, from the last time you painted it, that two people can paint the fence in eight hours; you further know that everyone who is available to help you paints at the same rate. Answer

- **55(a)** Is this a matter of inverse proportion? Answer
- **55(b)** If this is a matter of inverse proportion, what is the proportionality constant? Answer
- **55(c)** You're busy that day and have a maximum of four hours to complete the job. How many people will you need, including yourself, to finish in that time? Answer

56. The gravitational attraction of two objects is inversely proportional to the square of the distance between them. Answer

- **56(a)** Mars is about 2;8 astronomical units (au) away from Earth at its farthest point. Assume that its gravitational attraction of Earth at that distance is 1. What is its gravitational attraction of Earth at 2 au distance? Answer
- **56(b)** Our space station is 234 miles above Earth, and the gravitational attraction of Earth is 8Ɛ. If we increase our altitude to 2↊2 miles, what will the gravitational attraction be? Answer

6.3.4 Perbiquas

The last subject of ratios and proportions we must handle are *perbiquas*:

> ***perbiqua***
>
> any ratio expressed as parts per 100; from "per" and "biqua"; written as "%"

The etymology of the word is fairly clear. Perbiqua's decimal analog is the *percentage*:

> ***percentage***
>
> any ratio expressed as parts per 84, which is written as "100" in decimal; from "per" and "centum," the latter of which is the Latin word for "hundred"; written as "%"

100 is such a convenient value, with so many very useful divisions (it is divisible by 2, 3, 4, 6, 8, 9, 10, 14, 16, 20, 30, 40, and 60!), that we often find it useful to change ratios into an equivalent expression of parts per 100. For this reason perbiquas were devised.

To make it easier to write perbiquas, we use the symbol "%", and do *not* write the ratio as a vulgar fraction as we normally would. However, every perbiqua is a perfectly normal vulgar fraction, with 100 as the denominator. In other words, "46%" is equivalent to $\frac{46}{100}$, and "Ɛ6%" is equivalent to $\frac{Ɛ6}{100}$. Of course, this means that they're also equivalent to 0;46 and 0;Ɛ6, respectively.[17]

We make a perbiqua by using the technique of *cross-multiplying* that we learned in Section 6.3.2; we simply write our ratio, set it equal to some number (x) over 100, and then cross-multiply to solve. For example, to turn a ratio of 4 : 6 into a perbiqua:

$$\frac{4}{6} = \frac{x}{100}$$
$$6 \times x = 4 \times 100$$
$$6x = 400$$
$$\frac{6x}{6} = \frac{400}{6}$$
$$x = 80$$

That last answer tells us what the numerator in our perbiqua is; in other words,

$$\frac{x}{100} \rightarrow x = 80 \rightarrow \frac{80}{100}$$

[17] *See supra*, Section 1.5.2, at Ɛ.

So we're left with $\frac{80}{100}$. As we've seen, a perbiqua is just a vulgar fraction with a denominator of 100 which isn't written, plus the symbol "%" so that people know what we've done. So a 4 : 6 ratio is 80%.

Notice, as well, that any vulgar fraction with a denominator of 100 (or any other multiple of 10) can be easily changed to a digital fraction. So:

$$80\% = \frac{80}{100} = 0;80$$

So perbiquas can be turned into digital fractions simply by moving the dit two places to the left, and vice-versa by moving it two places to the right.

Exercises 6.14

57. Set the following ratios into perbiquas. Answer
57(a) 5 : 6 Answer **57(b)** 7 : 9 Answer **57(c)** 37 : 100 Answer
57(d) 2 : 3 Answer

58. Solve the following; if it is a digital fraction, convert it to a perbiqua, and vice versa. Answer
58(a) 0;47 Answer **58(b)** 63;21% Answer **58(c)** 1;439 Answer
58(d) 0;003 Answer

59. You are a teacher with 26 students. The last test you gave was passed by $\frac{5}{6}$ of those students. What is the pass rate for the test, expressed in perbiquas; and how many of your students passed? Answer

5↊. You are a school administrator and are told by state examiners that ↋0% of your students had passed the last state exam. Your school has 340 students who took that exam. How many passed it? Answer

6.4 Mean, Median, Mode, and Range

We first ran into some of these concepts in our study of square roots[18]; now, we will address them in earnest. These concepts are the very beginnings of the science called *statistics*, the study of large groups of numbers. You certainly won't be able to do anything but the most basic statistics with them; but they're necessary for every numerate person to know.

First, we need to note that these concepts are important when we're dealing with large groups of numbers. We will be dealing with smaller groups here for illustrative purposes, but remember that normally you'll be dealing with at least several dozen numbers, often called *data points*.

So let's take a smallish data set of six values:

8 49 ↊8 45 67 ↋3

Our first task is to find the *average*, or *mean*:

[18] *See supra*, Section 6.2.2.2, at 105.

mean

the quotient of the sum of a list of addends and the number of those addends

Fortunately, this is a simple process: as the definition of "mean" suggests, we just add them all up and then divide by their number. In this case, we calculate 8 + 49 + ↊8 + 45 + 67 + ƐƐ = 330, then divide that by 6 (since we added up six numbers): 330 ÷ 6 = 66. Our mean is 66.

Strictly speaking, there are many types of mean. For example, there is the *geometric mean*, where we would find the *product* of all the numbers and then take the root to the degree which equals the number of data points; or the *harmonic mean*, which is still more complex and still less often encountered. What we have learned here is the *arithmetic mean*, and it will suffice for our studies of arithmetic.

The arithmetic mean is designed to find the "central" or "typical" number in a set of numbers; in other words, it's a number that we can reasonably point to as characterizing the whole group. Occasionally, it will equal one of the data points themselves; however, normally it will *not*, so be aware that your mean will likely not be one of the numbers you used to obtain it.

However, the mean is only one way (and the most common way) of determining a "central" or "typical" number from a set; another is the *mode*:

mode

the most common value in a data set

So let's consider a slightly different set of numbers for our data set:

8 49 ↊8 45 49 67 ƐƐ

We have simply added a single data point to the list: the 49 between 45 and 67. All of these values occur only once in the data set, with the exception of 49, which occurs twice. Since it occurs more often than any other value, it is the *mode*.

Sometimes, no value will occur more often than any other, as in our original data set. In this case, there is no mode.

Sometimes again, more than one number may occur more than any other, but still be tied for most often. E.g., if we added another 45 to our data set, then 49 and 45 would both appear twice, which is still more than any other number. In this case, both are modes. (Some authors say that, in this case, there is no mode. You can use your judgment about this question.)

Notice that, unlike the mean, the mode (if it exists) will *always* be one of the values of your data set.

Finally, there is the *median*:

median
the central value of a data set, when that data set is enumerated from smallest to largest (or vice versa); if there is no such number because there is an even number of data points, the median is the mean between the two central data points

Notice that, to determine the median, we may need to reorder our data set. Consider our modified data set:

$$8 \qquad 49 \qquad 82 \qquad 45 \qquad 49 \qquad 67 \qquad 33$$

We can't determine a median for this data set because it is not in order; if it's not in order, we can't see which one is in the middle. So let's fix that:

$$8 \qquad 45 \qquad 49 \qquad 49 \qquad 67 \qquad 82 \qquad 33$$

Now the data set is ordered in value from smallest to largest, and we have highlighted the central value in blue. This value is the median.

The student will notice, however, that this method only works if there is an odd number of data points; with an even number, there won't be a middle value. To illustrate, take our original data set again:

$$8 \qquad 45 \qquad 49 \qquad 67 \qquad 82 \qquad 33$$

(We've reordered it for purposes of locating a median.) But what's the middle value? As the coloring makes clear, there simply isn't one; we have the same number of values on both sides, and no extra one left over to be our middle. We do, however, have two which are equally middle values:

$$8 \qquad 45 \qquad 49 \qquad 67 \qquad 82 \qquad 33$$

So we take the mean of the two which are closest to the middle, and call it a day. $(49 + 67) \div 2 = 58$. 58 is, then, our median.

Given this procedure, there will *always* be a median value. If you have an odd number of data points, the median will equal one of your data points; if you have an even number of data points, it will not.

Finally, we have a data set's *range*:

range
the difference between the largest and smallest data points in a set

Unlike mean, median, and mode, the range is not an attempt to describe the "typical" number in a set; rather, it is a description of how far spread out the data points are.

While it doesn't describe the "normal" value of the set, it does give you an idea of the different values of the set, and thus is valuable in its own right.

Exercises 6.15

5Ɛ. You are a teacher with an unusually small class, and have just graded your students' final exams. The grades are 78, 98, Ɛ3, Ꮭ5, 34, Ꮭ8, Ꮭ5, and Ꮭ8. Give the mean, median, mode, and range of the grades. Answer

60. One of the students in your class received the following grades: 56; 66; 86; 80; 96; Ꮭ5; Ꮭ8. Give the mean, median, mode, and range of his grades; round or truncate to four places, if necessary. Answer

Part III
Mental Arithmetic

Chapter 7
Basics of Mental Arithmetic

Too many students believe that once they have mastered written arithmetic, they have mastered the entirety of the discipline. And indeed, mastering the written algorithms of arithmetic is the most important step. But true mastery of arithmetic can only come with facility in *mental* arithmetic: performing arithmetical operations without putting pen to paper (or finger to button, as the case may be).

Mental arithmetic is important for a number of reasons. First, nothing internalizes the algorithms of arithmetic like learning how to utilize them in the mind. Second, the practice of mental arithmetic requires facility in breaking up problems into smaller parts, which illuminates the nature of number and the workings of our arithmetical algorithms. Finally, we won't always have calculators or pen-and-paper handy; and even when we do, it is often quicker and easier to do the calculations mentally, provided we know how.

There is no substitute to knowing the real, full algorithms we learned in Part II; but learning to apply these in mental arithmetic is a useful and important exercise.

Also, it is worth mentioning, many of these methods can be used to simplify problems even when you're using the written algorithms, particularly those involving division.[1] So even a student uninterested in mental arithmetic would do well to study this chapter.

7.1 The Fundamental Rules of Mental Arithmetic

We must examine first the three fundamental rules of mental arithmetic, rules which will inform nearly every one of the methods that we'll explore here.

Rule of Elementary Results

Every operation in mental arithmetic depends on knowing elementary results.

Beginning mental arithmetic requires thorough memorization of *elementary results*, without which it's impossible to get even a start. Fortunately, we've already spent enough time with arithmetic that the student likely knows most, if not all, of these already; but a review will not be unwelcome.

elementary result

a result which can be found on the addition or multiplication table, as the result of an addition, subtraction, multiplication, or division problem

We have already met the addition table[2] and multiplication table[3]; we can also find them, among other helpful tables, in Appendix E.[4] These should be committed to memory

[1] *See infra*, Section 7.5, at 13Ɛ. [2] *See supra*, Section 5.1.2, at 39. [3] *See supra*, Section 5.3.2, at 55.
[4] *See infra*, at 165.

as firmly as possible. One need not do so by rote; that is, by simply reciting the table until one can do so without looking. It is much more advantageous to learn individual facts from these tables, along with a few simple rules.

Flash cards are also an excellent way to get accustomed to these elementary results. Electronic aids are available, which we will discuss in Appendix B.

Ultimately, though, the solution is just *lots of practice*.

There are two other general rules which will come in handy as you learn methods of mental arithmetic.

Rule of Opposite Direction

Most mental arithmetic is done more easily from left to right, rather than from right to left, as in the written algorithms.

This principle is best learned by demonstration, and we'll be seeing how well it works in various situations as we encounter them. However, it's important to know that this is a general principle in working mental arithmetic.

Rule of Problem Division

Mental arithmetic involves breaking single, complex problems into multiple, simple ones.

We've done this sort of thing many times in our journey through arithmetic thus far, particularly when we've learned advanced arithmetic[5]; however, in mental arithmetic even problems that we attacked directly, such as multiplication of two-digit numbers, must be divided into multiple parts to be easily handled.

Remember these three rules as you go through the reading in the following sections; you'll encounter them again and again, and see how truly fundamental they are.

7.2 Mental Addition

We've already memorized all the one-digit sums there are; with this knowledge, summing multiple-digit numbers is just a short step. The trick is to remember that addition is commutative and associative, so we can break the numbers apart and add the pieces together however we want.

Let's consider the problem $45 + 34$; this problem will be an easy one because it does not involve carrying. We have already seen how to set this up for pen-and-paper work with our addition algorithm:

$$\begin{array}{r} 4\;5 \\ +\,3\;4 \\ \hline \end{array}$$

[5] *See supra*, Chapter 6, at 79.

Adding 45 to 34 is tough all at once; but we know that with our pen-and-paper algorithm, that's not really what we're doing. What we're really doing is adding 5 and 4 first, then adding 40 and 30, and putting them together.

So do that instead! Thanks to our memorization of elementary results, we already know that $5 + 4 = 9$; we also know that $4 + 3 = 7$, so $40 + 30 = 70$. That gives us our answer: $45 + 34 = 79$.

One of the chief methods of mental arithmetic is *breaking complex problems into simpler ones*. By this means, addition problems of significant complexity are made quite simple. Consider $45 + 39$ now. We break the problem up, adding the unquas first and the ones later. So to add the unquas, we have $4 + 3 = 7$, so $40 + 30 = 70$, just as before. Now we add the ones: $5 + 9 = 12$, an elementary result. But 12 means that we have one more unqua, so we increase 70 to 80, then add the 2 from our 12 to give an answer of 82. $45 + 39 = 82$.

So try this for a while; rather than adding 45 and 39 all at once, add 40 and 30, then 5 and 9, and total it. Although this method gives more steps, each step is much easier, so you'll be able to solve these problems mentally quite quickly.

You can do the same thing with problems of many more digits; just keep track of each digit separately, and as long as there aren't too many digits to track reliably, you'll get the right answer.

When carrying is involved, we often can do the same thing, particularly with smaller numbers; just divide the problem up in a different way. Consider the following problem:

$$\begin{array}{r} 2\ 8 \\ +3\ 7 \\ \hline \end{array}$$

We'd have to carry to do this problem on paper, which means that our last method would be difficult. So let's take one number as a whole, then add *only the ones digit of the other number* to it. This will be easy:

$$28 + 7 = 33$$

This is really just a matter of doing an elementary result, plus tacking on another one to the unqua digit, an easy problem for us by now. We've added 7 to 28; now we're left with 30 we still need to add:

$$\begin{array}{r} 3\ 3 \\ +3\ 0 \\ \hline \end{array}$$

But that's easy, too; we're still just adding one digit at this point, the unqua digit; $3 + 3 = 6$. We still need the ones digit; but that's easy, too, because now we're adding zero, which doesn't change it at all. so $28 + 37 = 63$.

A final method, often helpful with larger numbers, is important enough to become our second cardinal rule of mental arithmetic: *solve the problem from* left to right*, rather than from right to left*, as we do on paper. Let's consider one such problem:

$$\begin{array}{r} 4\ \text{↊}\ 5 \\ +\,7\ 4\ 7 \\ \hline \end{array}$$

Clearly, we're going to have to carry at every digit here; but if we try to keep track of all the carries while we also try to keep track of our sum, before long we'll lose track of everything. Instead, add them up from the left side.

First, sum 4 and 7, giving ↋. Easy enough; you have ↋ biqua.

Next, add the unqua digits, ↊ and 4. This makes 12. You already know that you've got ↋ biqua; now you know that you have 12 unqua. But you can't fit 12 into a single digit; you've got to carry that 1 to the biqua digit. Your biqua digit is ↋; so add the 1 from the 12 to it, giving you 10, and then put in your 2, giving you 102? (the "?" showing that you still don't have a ones digit figured out yet).

Now add 5 and 7, which of course make 10. Again, you can't fit 10 into a single digit, so you'll have to put the 1 in the unqua digit and the 0 in the ones digit. Your unqua digit is 2; so add the 1 to it to make 3, and your ones digit is 0. Now you know your sum: 1030.

Did it work? Let's check:

$$\begin{array}{r} {\scriptstyle 1\ \ 1\ \ \ } \\ 4\ \text{↊}\ 5 \\ +\,7\ 4\ 7 \\ \hline 1\ 0\ 3\ 0 \end{array}$$

It did!

You're performing the same algorithm—just adding each digit and carrying the surplus over to the next—but you're doing both at the same time, so that you only need to keep track of one set of numbers—the digits of your sum—instead of two—the digits of your sum as well as your carries.

As always with mental arithmetic, practice! It gets much easier the more you do.

7.3 Mental Subtraction

Subtraction is trickier for larger numbers, but as long as we remember our prime mental arithmetic tactics—dividing a big problem into a few very small problems, and solving a problem from left to right—we'll still get the trick. We need to be more careful, because subtraction is *anticommutative*; we can't divide problems up willy-nilly, but have to be careful about the order in which we do it. However, in this section, we'll learn to safely divide difficult subtraction problems into simple addition problems.

We've already seen that addition and subtraction are just two sides of the same coin[6]; now we can take advantage of that to solve problems mentally. Consider subtracting 17 from 25, for example. We'd set this up as follows for our pen-and-paper algorithm:

$$\begin{array}{r} 2\ 5 \\ -\,1\ 7 \\ \hline \end{array}$$

To solve this mentally, think instead, *how many must I add to 17 to get to 25?* And then take it step by step. To get to 20 from 17, I add 5 (since $7 + 5 = 10$, this step is easy). Then, to get from 20 to 25, I add 5 more (since $0 + 5 = 5$ is also easy). So I take the first 5 I added and add it to the second 5, giving me ↊. $25 - 17 = \text{↊}$.

Notice that we managed to do that problem, with only two addition problems, *with no borrowing*, even though our paper algorithm would have required borrowing.

Let's try a more difficult problem, where the difference is more than an unqua.

$$\begin{array}{r} \text{↊}\ 8 \\ -\,4\ 9 \\ \hline \end{array}$$

Here, we do the same thing as before, but we track first how many *unqua* we have to add to 49 to equal ↊0, and then proceed to see how many ones we need to reach ↊8.

1. How many unquas must be add to 49 to make ↊0? $\text{↊} - 4 = 6$, so we need 6 unquas.
2. 49 plus 6 unquas $(49 + 60)$ is ↊9.
3. How many ones do we need to get from ↊9 to ↊8? The answer, of course, is -1.
4. $60 + -1 = 5\text{↋}$.

Like addition, we can extend this process for as many digits as we can keep track of in our minds.

7.4 Mental Multiplication

Multiplication is more difficult than addition or subtraction even on paper, and it is still more difficult mentally. Still, as long as we keep in mind our fundamental rule for mental arithmetic—separate large, complex problems into smaller, simpler ones—usually we can manage it.

[6] *See supra*, Section 5.2.6, at 51.

7.4.1 Single-Digit Multipliers

We'll begin by taking the simplest case (besides, of course, the elementary results): multiplying by a one-digit multiplier. Our example:

$$\begin{array}{rrr} & 8 & 3 \\ \times & & 9 \\ \hline \end{array}$$

By now, we know the basic drill: moving left to right, multiply first 8 by 9, then 3 by 9, then add the products. However, we have to be careful: when we multiply an unqua by a one, as we are when we multiply 8 by 9 here, *we need to remember to add a zero at the end*; otherwise, we're putting in number that's an order of magnitude too small.

$8 \times 9 = 60$ and $3 \times 9 = 23$, as we already know. Since one of the two factors in the latter problem is an unqua (the 8 really represents 80), we need to add a zero to that product: 600. Then we sum the two: $600 + 23 = 623$.

But does it really work?

$$\begin{array}{rrr} & \overset{2}{8} & 3 \\ \times & & 9 \\ \hline 6 & 2 & 3 \end{array}$$

It does! This works for any problem with a one-digit multiplier; just remember that for each order of magnitude, you need to add another zero to your intermediate product. Let's take another example:

$$\begin{array}{rrrr} & 7 & 8 & 3 \\ \times & & & 9 \\ \hline \end{array}$$

Try calling out the numbers as you do the problem in your head. Start on the left. $7 \times 9 = 53$, you can start by calling out, "53" (knowing that this is really 5300; since you're multiplying a biqua, you need to add two zeroes). Then you do the next digit; $8 \times 9 = 60$, but you know this is really 600; since we're multiplying an unqua, we need to add one zero. So you know you're adding 5300 and 600. But you've already called out "53"; say, "No, 59," and move on.

Next, you multiply the final digit. $3 \times 9 = 23$, of course, and in this case it really does mean just 23, with no zeroes added. So add it to your 5900, and call out, "5923."

As you can see, multiplying multiple digits by one digit is fairly simple; we'll see some harder problems as we move along.

7.4.2 Multiplying Larger Numbers

There are many methods of multiplying larger numbers, some of which are quite specific to certain types of numbers. We'll look at a handful of these here.

7.4.3 Multiplying by Multiples of 10

We saw briefly, in Section 5.3.3,[7] that multiplying by any number ending in 0 is easier than multiplying by other numbers. Here's, we'll examine that principle a little more closely.

Take the following example:

$$\begin{array}{r} 4\ 7 \\ \times\ 4\ 0 \\ \hline \end{array}$$

As any student at this point will immediately notice, the multiplier here is a multiple of 10. This gives it special properties that make it especially easy to multiply by. We know, again from Section 5.3.3,[8] that when we are multiplying by a 1 followed by any number of zeroes, we simply move the dit of our multiplicand that number of places to the right. So, for example, $4;738 \times 100 = 473;8$, or $4;7 \times 1000 = 4700$.

We briefly saw how to take that one step farther, with numbers other than 1 followed by some number of zeroes. We are now faced with such a problem here.

So, following the cardinal principle of breaking up difficult problems into multiple easy ones, let's do the same here. It's difficult to multiply two two-digit numbers; however, it's easy to multiply a two-digit number by a one-digit number, and it's easy to multiply by 10. So let's multiply 47 by 4, then multiply that product by 10.

We can divide either factor however we want; because multiplication is commutative, it doesn't matter what order we put the various factors in. This means that it's perfectly fine to break up either factor into as many of its own factors as will make the problem easier to work. Here, we're breaking up the multiplier into two of its factors, 4 and 10, and taking the problem from there.

$47 \times 4 = 164$, and $164 \times 10 = 1640$; therefore, $47 \times 40 = 1640$. Do the problem the long way if you don't believe it; but it works, every time.

You can also use this trick for numbers *near* a multiple of 10. For example, Ɛ. Just round up to the multiple of 10, solve, then subtract the other number once. We'll see this method generalized in Section 7.4.5.[9]

7.4.4 Doubling and Halving

We can use doubling and halving most easily whenever the multiplier is an even number. Because multiplication is commutative, we can halve the multiplier and double the

[7] *See supra*, at 56. [8] *See supra*, at 56. [9] *See infra*, at 139.

multiplicand in order to turn a complex, multi-digit-multiplier problem into a simple one-digit-multiplier problem.

$$\begin{array}{r} ↊\ 3 \\ \times\ 4\ 8 \\ \hline \end{array}$$

Notice that our multiplier here is *even*. So let's take advantage of that. If we halve the multiplier and double the multiplicand, we will end up with not ↊3 × 48, but 186 × 24. Notice that our multiplier is still even; let's do it again: 350 × 12. And again: 6↊0 × 7. And now we have a problem with a one-digit multiplier, and we well know how to do that:

- 0 × 7 = 0
- ↊ × 7 = 5↊
- 6 × 7 = 36
- Add: 3600 + 5↊0 + 0 = 3↋↊0

Multiplying and dividing by 2 is comparatively easy (that is, compared to multiplying by 48, which is what we started with); so even though we're doing many more operations, we're doing much easier ones, which makes the problem as a whole mentally manageable.

This works just as well with other factors; for example, perhaps your multiplier isn't even, but is divisible by 3. Then, if you can, triple your multiplicand and third your multiplier. But doubling and halving is by far the easiest.

If we have an *odd* divisor, we need to do a little more work. Let's slightly modify our problem:

$$\begin{array}{r} ↊\ 3 \\ \times\ 4\ 9 \\ \hline \end{array}$$

Now, our multiplier is odd, so our doubling and halving trick won't work. So just make it even! Remember, though, that you have done so. Instead of multiplying ↊3 by 49, multiply ↊3 by 48 as before, giving you your answer of 3↋↊0. Now, however, remember that you took one off your multiplier to make it even; this means that you must add one more ↊3 to the answer to make it work. 3↋↊0 + ↊3 = 4083.

No one would pretend that even this method is *easy*; but it's easi*er* than long multiplication, particularly when done mentally, and with some practice, you'll be pretty impressive juggling such figures in your head.

It's also not very helpful if you can't halve the multiplier until you get to a low enough number. But this will work for a lot of problems.

7.4.5 Breaking the Factors

Another technique, more general than doubleing and halving,[↊], involves *breaking the factors*.

We can break whichever of the factors will lead to an easier problem; whether it's the multiplicand or the multiplier, the procedure is the same. Let's take our sample problem yet again:

$$\begin{array}{r} ↊\,3 \\ \times\,4\,9 \\ \hline \end{array}$$

We notice that both of these factors are the same distance from its nearest multiple of twelve; ↊3 is three away from ↊0, and 49 is three away from 50. So it's really of no moment which of the factors we select to break; for demonstration's sake, let's select the multiplier, 49.

Break it; that is, change it to its nearest multiple of 10, and keep in mind how much you had to change it. In this case, we *increased* it, by 3. Store that 3 away in your mind for the moment. Your mental problem is now this:

$$\begin{array}{r} ↊\,3 \\ \times\,5\,0 \\ \hline \end{array}$$

Now, *that*'s easier to deal with; multiplying by an even unqua is much easier than multiplying by 49. Multiply ↊3 by 5, then add a 0 to the product. We're well aware of the speedy methods to multiply by a one-digit multiplier,[↋] so this shouldn't be too difficult:

$$\begin{array}{r} ↊\,3 \\ \times\quad 5 \\ \hline 4\,3\,3 \end{array}$$

Then add a 0; so our interim product is 4330. Now, remember that 3 that we stored away a while ago? Multiply that by the factor that we *didn't* break; in this case, ↊3:

$$\begin{array}{r} ↊\,3 \\ \times\quad 3 \\ \hline 2\,6\,9 \end{array}$$

[↊] *See supra*, Section 7.4.4, at 137. [↋] *See supra*, Section 7.4.1, at 136.

Now, since we *increased* the size of one of the factors, we *subtract* this 269 from our interim product, 4330. (If we'd *decreased* a factor, we'd then *add* it.) Again, this is a type of problem we're well able to do mentally:

$$\begin{array}{r} 4\ 3\ 3\ 0 \\ -\ 2\ 6\ 9 \\ \hline 4\ 0\ 8\ 3 \end{array}$$

And that's our answer: ↊3 × 49 = 4083.

Once again, we're forced to admit that what we're doing here isn't simply *easy*; it's actually quite difficult, and one must have a very firm grasp of the algorithms of arithmetic to make sense of it. But with that knowledge, and with some practice putting that knowledge into use, one can solve these problems quite quickly.

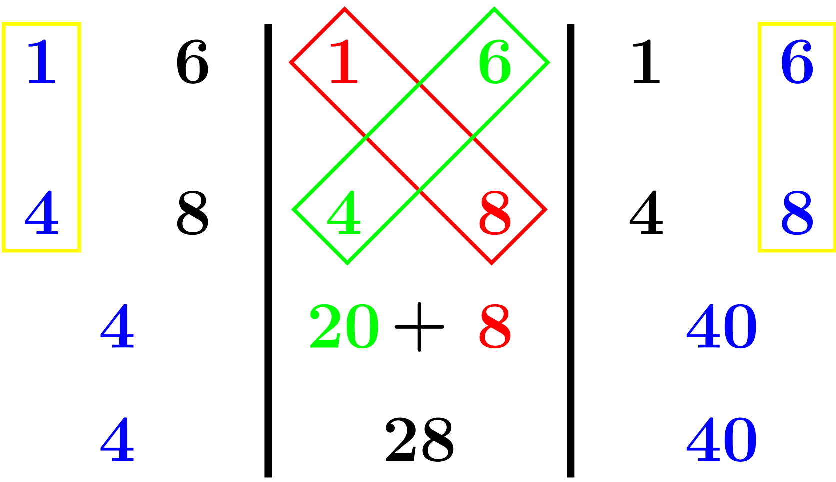
7.4.6 Singling and Crossing

Another handy method of mental multiplication is *singling and crossing*. This will work on factors of any length, but is by far easiest when both factors have two digits.

Let's take an example problem:

$$\begin{array}{r} 1\ 6 \\ \times\quad 4\ 8 \\ \hline \end{array}$$

Now, rather than multiplying in a nice, orderly way, as we do in our written algorithm, we'll multiply it a bit chaotically. First, multiply the unqua digits. Then, *cross-multiply*, as if you're working with a proportion; multiply the unqua digit of the multiplicand with the one digit of the multiplier, and vice-versa. Finally, multiply the two ones digits together. Put them together to get the answer.



Biquas	$\mathbf{4 \times 100}$	**400**
Unquas	$\mathbf{28 \times 10}$	**280**
Ones	$\mathbf{40 \times 1}$	**40**
		700

As the student will see, multiplying the unquas of the two factors yields the biqua digit (note that, if this result is 10 or greater, there will be a triqua digit, as well); cross-multiplying will yield the unqua digit; and multiplying the ones of the two factors yields the ones digit. Because frequently one or more of these will have multiple digits, we must adjust them by adding them together, as shown in the table above.

This is *much* easier to do mentally than our full written algorithm, and is one of the easiest methods for problems of this type.

7.5 Mental Division

As we're all used to by this point, division is the most difficult operation to do mentally. Our written algorithm is so complex that it should come as little surprise to anyone that the mental methods are at least equally so.

Nevertheless, there are ways to do it, and often to do it easily. We'll explore a few here.

7.5.1 Single-Digit Divisors

We will learn some ways to do this type of problem more quickly and easily, of course; but the general method of dividing mentally should be explored first.

Fortunately, we *already* divide left-to-right, even in our written algorithms (namely, long division and short division); so the process is pretty intuitive for us at this point. Indeed, what we do mentally is, essentially, short division, so we already know what to do.

$$8 \overline{)\,6\ 5\ 4}$$

At first glance, we can tell that this will come out evenly, because 654 is clearly divisible by 8 (assuming that we've memorized our tables in Appendix E, anyway); now we need to do the actual division. So, starting with the first digit of the dividend, we see how many of our divisor are contained in it. Here, 0; so we grab another digit of our divisor and check that. 65 contains 8 9 times, with 5 left over; so we know that the first digit of our quotient is 9. We then use that 5 and join it with the next digit of our dividend, giving us 54, which we know divides evenly to 8 (because it's an elementary result). So our answer is 98.

$$\begin{array}{r} 9\,8 \\ 8\,\overline{)\,6\,5\,4} \end{array}$$

But even this problem could be simplified to be done much more quickly and easily; so after we've done a few exercises, let's take a look at a few methods of doing so.

7.5.2 Simplify As Far As Possible

We noticed, when we were working with fractions, that any fraction can be *reduced*,[10] and we further noticed that fractions are just division problems.[11] Remembering that, we can frequently reduce division problems into much simpler ones.

To reduce a fraction, we just perform the same division operation on both the numerator and denominator; to simplify a division problem, we can do the same division operation on both the dividend and the divisor. As we know from our work with fractions, this will not change the final answer; but it may make the problem much easier to work with. Let's take an example:

$$↊6\,\overline{)\,4\,4\,6}$$

We can significantly simplify this problem simply by dividing both its dividend and its divisor by the same number. Looking at the two of them, they are both divisible by 6 (as we know, if we've learned all our tables in Appendix E); so let's divide both by 6 and see if that problem is easier:

$$19\,\overline{)\,8\,9}$$

Now we have a much shorter problem. But, looking at these new numbers, they are both divisible by 3 (again, remember your tables in Appendix E); so let's divide them by 3 and see if that further facilitates our operation:

$$7\,\overline{)\,2\,↋}$$

And now, we definitely know the answer: $2↋ \div 7 = 5$. To check it, you can always try the original problem; you will find that $446 \div ↊6$ also equals 5. We turned a very complicated long division problem into a nice, easy elementary result, simply by reducing it in a direct, orderly way.

[10] *See supra*, Section 5.5.1.5, at 94. [11] *See supra*, Section 5.5.1.4, at 92.

7.5.3 Multiply Before Dividing

While it may seem counterintuitive, sometimes the easiest thing to do is to *multiply* before you *divide*.

Remember that, with division, we'll always get the same answer as long as we do the same thing to both the dividend and the divisor. So sometimes it's helpful to turn one of these terms—most often, the divisor—into an easier-to-handle number by multiplying both of the terms first.

For example:

$$6\overline{)336}$$

Dividing by 6 is hard; at least, harder than dividing by 10. But dividing by 10 is easy; just move the dit! So let's turn this into a division-by-10 problem by multiplying both terms by 2:

$$10\overline{)670}$$

Multiplying by 2 is also quite easy, so this shouldn't be a problem. And now the question is easy; $670 \div 2 = 67$. And we can check it against our original problem by doing the long division: $336 \div 6 = 67$. It works!

It works with other problems, too. Perhaps we're dividing by 4:

4)8↊4

We can tell at first glance that it will come out evenly (since 8↊4 is divisible by 4, as we know from our tables in Appendix E); but it's still a bit difficult to do directly. So let's make it a division-by-10 problem by multiplying both sides by 3:

$$10\overline{)2270}$$

And now we can easily get the answer, 227.

Sometimes, *dividing* first is a better choice. Perhaps we're again facing the following division problem, and we want to approach it a different way:

4)8↊4

Perhaps we're more comfortable dividing by 2 than by 4. So make it a division-by-two problem, by dividing both terms by 2:

$$2\,\overline{)\,4\;5\;2}$$

Now you have a problem you're more comfortable doing, and you can easily derive the answer, 227.

7.5.4 Division of a Two-Digit Number by ↋

Division by ↋ is surprisingly simple, when the dividend is a two-digit number. Such problems follow two rules:

1. If the sum of the digits is *less than ↋*, the quotient is the unqua digit of the dividend, and the remainder is the sum of the digits.
2. If the sum of the digits is *↋ or more*, the quotient is the unqua digit of the dividend plus 1, and the remainder is the sum of the digits mod ↋.

Let's take an example of the first rule:

$$↋\,\overline{)\,5\;2}$$

First, we sum the digits of the dividend; $5 + 2 = 7$, and $7 < ↋$. So we know that we're in the realm of the first rule, since the sum of the digits is less than ↋. That means that the quotient is 5. The remainder is the sum of the digits, or 7.

Now, let's try a slightly different problem:

$$↋\,\overline{)\,5\;↊}$$

This time, when we sum the digits, we get $5 + ↊ = 13$, and $13 > ↋$. So we know that we're in the realm of the second rule. So the quotient is the unqua digit of the dividend plus 1, or $5 + 1$; the quotient is 6. The remainder is the result of $(5 + ↊)$ % ↋, or 13 % ↋; practically, this resolves to $13 - ↋$, or 4. So the answer is 6 R 4.

While this only works for two-digit dividends, it's still a very handy trick when the occasion for it arises.

Appendices

Appendix A
Answers to the Exercises

Exercises from Chapter 1

Exercises 1.1 **1.** **1(a)** 913 **1(b)** 415; remember that, after counting 10 of a group, you must move one more into the next group and start counting again! **1(c)** 50↊; when there are no groups for a digit, put in 0. **1(d)** 46; really 046, but any number of zeroes on the left can be left out. That's because "046" and "46" are still the same number. **1(e)** 1340; remember that, after counting 10 of a group, you must move one more into the next group and start counting again! Remember also that you can't leave out a 0 in the middle or end of a number; otherwise our number here would look like 134, a very different number than 1340.

Exercises 1.2 **2.** **2(a)** Natural, whole, or real. **2(b)** Natural, whole, or real. **2(c)** Real. **2(d)** Whole or real. **2(e)** Real. **3.** **3(a)** Rational, nonrepeating. **3(b)** Rational, repeating. **3(c)** Irrational. **3(d)** Irrational. **3(e)** Rational, nonrepeating.

Exercises 1.3 **4.** 43, 48, 51, 56, 5Ɛ, 64, 69, 72, 77. **5.** 32, 35, 38, 3Ɛ, 42, 45, 48, 4Ɛ, 52, 55, 58, 5Ɛ, 62, 65, 68, 6Ɛ, 72, 75, 78, 7Ɛ, 82, 85, 88, 8Ɛ, 92, 95. **6.** 0, 2, 4, 6, 8. (Note that 9 is not in the sequence, so we don't count it.) **7.** 34, 40, 48, 54. **8.** 50, 60, 70, 80, 90, ↊0. **9.** 3, 7, Ɛ, 13, 17, 1Ɛ, 23, 27, 2Ɛ, 33, 37, 3Ɛ, 43, 47. **↊.** 43, 53, 63, 73, 83, 93, ↊3, Ɛ3.

Exercises from Chapter 2

Exercises 2.1 **1.** **1(a)** Nine. **1(b)** One unqua four (onequa four). **1(c)** Seven unqua eight (sevenqua eight). **1(d)** Ten unqua two (tenqua two). **1(e)** Four biqua nine seven. **1(f)** Eleven biqua three eight. **1(g)** Six triqua five two ten. **1(h)** Three. **1(i)** Eight ennqua eleven six nine three zero three ten zero. **1(j)** Five hexqua four three five ten zero two. **1(k)** Six pentqua eleven four zero three six.

Exercises 2.2 **2.** **2(a)** 9×10^{8}. **2(b)** Ɛ $\times 10^{-8}$. **2(c)** ↊;8425 $\times 10^{-1}$. **2(d)** 9;05Ɛ↊ 492↊ 386 $\times 10^{4}$ **3.** **3(a)** $^{8}9$. **3(b)** $_{8}$Ɛ. **3(c)** $_{1}$↊;8425. **3(d)** ^{4}9;05Ɛ↊ 492↊ 386 **4.** **4(a)** ↊0000 **4(b)** 6 0000 0000 **4(c)** 0;0000 08Ɛ34↊ **4(d)** 0;00084

Exercises 2.3 **5.** **5(a)** *Quadrahedron* **5(b)** *Hexahedron* ("cube" is, of course, still a useful English word) **5(c)** *Octahedron* **5(d)** *Unquahedron*; *unnilihedron* **5(e)** *Unoctahedron* **6.** **6(a)** *Biweekly*; *dithexmonthly*; *unperbimonthly*. **6(b)** *Dittriyearly*; *trimonthly*; *unperquadyearly*. **6(c)** *Triweekly*. **6(d)** *Ditquadyearly*; *quadmonthly*; *unpertriyearly*.

Exercises from Chapter 3

Exercises 3.1 **1.** **1(a)** IV **1(b)** XIV **1(c)** XIX **1(d)** XXVI **1(e)** XXIV **1(f)** XXXIII **1(g)** CCXL **1(h)** CCCXXXIX **1(i)** MDCCCXXVIII

1(j) MMXVI **2.** **2(a)** 54 **2(b)** 54 **2(c)** 2048 **2(d)** 1024 **2(e)** 1492 **2(f)** 144 **2(g)** 1492 **2(h)** 144

Exercises 3.2 **3.** **3(a)** IIV **3(b)** XIIV **3(c)** XVIII **3(d)** XXV **3(e)** XXIIV **3(f)** XXXIII **3(g)** CCXXL **3(h)** CCCXXXVIII **3(i)** MDCCXXVII **3(j)** MMXV **4.** **4(a)** 63 **4(b)** 65 **4(c)** 2059 **4(d)** 1025 **4(e)** ↊44 **4(f)** 155 **4(g)** 1572 **4(h)** 144

Exercises from Chapter 4

Exercises 4.1 **1.** **1(a)** The order of the operands doesn't matter. **1(b)** The order of the operations doesn't matter. **1(c)** Reversing the operands produces the negative result. **1(d)** An operation where each individual input produces one and only one output. **2.** **2(a)** Anticommutativity **2(b)** Function **2(c)** Commutativity **2(d)** Associativity

Exercises from Chapter 5

Exercises 5.1 **1.** 11 **2.** ↋ **3.** 4 **4.** 9 **5.** 9 (remember that addition is commutative!) **6.** 13 **7.** 12 **8.** 12 **9.** ↋ **↊.** ↋ **↋.** ↋ **10.** ↋

Exercises 5.2 **11.** 1 41↋9 **12.** ↊023 **13.** 5 2978 **14.** 769↋ **15.** 1 7851 **16.** 12 9569 **17.** 1027↊8 **18.** 2994 **19.** 1↊077 **1↊.** 147;↊1 **1↋.** 1↊7;4↋ **20.** 21↊;6 **21.** 5504;3568 **22.** 76911;2306 Notice that the third number, 249, has no radix point; so we line it up as if the radix point is on the far right. **23.** **23(a)** $↊;00. **23(b)** $14;60. **23(c)** $12;00. **24.** **24(a)** 114;↊ and 116;4↋; Racer #1. **24(b)** 114;49 and 119;03; Racer #3. **24(c)** 10↋;↋9 and 10↊;74; Racer #6. **24(d)** Racer #6.

Exercises 5.3 **25.** 4444 **26.** 1↊1↊ **27.** 902↋ **28.** 415↋;5↊8 **29.** 4005;↊8↋9 **2↊.** $↊9. **2↋.** **2↋(a)** $3↋7. **2↋(b)** 5 days longer. **2↋(c)** No "right" answer; decide for yourself.

Exercises 5.4 **30.** −486;2↊7 Switch the numbers; 28;495 + −4↋2;78. Two unlike signs, so it's really a subtraction problem; 28;495 − 4↋2;78. However, the minuend must be larger than the subtrahend, so switch them back; 4↋2;78 − 28;495 = 486;2↊7. However, we switched a subtraction problem, so negate the answer; −486;2↊7. **31.** −51↋;055 Switch the numbers to put the signs next to each other; 28;495 − −4↋2;78. Two like signs, which makes an addition; 28;495 + 4↋2;78 = 51↋;055. However, we reversed a subtraction problem, so negate the answer; −51↋;055. **32.** −0;697 Remember to switch the numbers around; subtract; then, since you switched a subtraction problem, negate the answer. **33.** −6;205 Two unlike signs, so it's really a subtraction problem; −2;975 − 3;45. Switch them; 3;45 − −2;975. Two like signs, so addition; 3;45 + 2;975 = 6;205. But we reversed a subtraction problem, so negate that; −6;205. **34.** 0;697 Two like signs, so really addition; −2;975 + 3;45. Switch; 3;45 + −2;975. Two unlike signs, so subtraction; 3;45 − 2;975 = 0;697. But we reversed a subtraction, so negate it; −0;697. **35.** −0;697. Two unlike signs, so really subtraction; this makes it the same as Example 32. **36.** **36(a)** $2↋50. This is a simple and straightforward addition problem. **36(b)** $750. A simple and straightforward subtraction problem; find the *difference* between total income and total outlays, using the larger number as the minuend. **36(c)** $−750. Like the previous question, except that you start with the income as minuend and expenditures as subtrahend. **36(d)** Assuming that your estimate of increased income is accurate, yes. Add $100 (the cost of the ad) to your monthly expenditures, giving 2↋50 + 100 = 3050. Now add $900 to your monthly income, giving 2400 + 900 = 3100. 3100 > 3050 (barely), so you've put yourself back in the black! **36(e)** Simple difference problems. Assuming the lower threshhold: 2↋50 + 100 = 3050 in monthly expenditures; this will be constant in both calculations. 2400 + 820 = 3020; 3020 − 3050 = −30, and you're still in the red, albeit less so. Assuming the higher threshhold: 2400 + 1030 = 3430; 3430 − 3050 = 3↊0, and you're much more comfortably in profit range.

Exercises 5.5 **37.** 20. **38.** 2↋. **39.** 34. **3↊.** 76. **3↋.** 34. **40.** 60. **41.** 36. **42.** 21. **43.** 41. **44.** 24. **45.** 24. **46.** 23. **47.** 69. **48.** 92. **49.** 47. **4↊.** 39.

Exercises 5.6 **4↋.** 0. **50.** 7389. **51.** 73890. **52.** 738;9. **53.** 738900. **54.** 7389. **55.** 738;9. **56.** 0.

Exercises 5.7 **57.** 3↋728 **58.** 4082 **59.** 228 **5↊.** 429↋716 **5↋.** 59440↋ **60.** 4134 7899 **61.** 2504 9642 80↋4 **62.** 92↊0 2386. With multiplication, it's easier to break the problem up; first do 3979 × 45 = 14 9729, then do 149729 × 672 = 92↊0 2386. Remember too that multiplication is commutative; you can do these in whatever order you like and get the same answer. **63.** 1139 ↊4↋5 1↋90 **64.** 7 03↊3 84↊8. Remember that multiplication is commutative. Our algorithm is much easier when the number with the fewer digits is on the bottom; so just switch them around and do them that way. 24 × 39↋4 is the same as 39↋4 × 24, and equals 8↋254; then we can do 8↋254 × 9532 = 7 03↊3 84↊8. The order in which the numbers are multiplied makes no difference in the answer. **65.** The problem is 164 × 46, yielding $6↊60. **66.** **66(a)** The problem is 14 × 3 = 40 gallons of General Ts'o sauce, and 13 × 3 = 39 gallons of sweet and sour sauce. **66(b)** You need 40 × 13 = 500 pounds of chicken and 39 × 13 = 483 pounds of pork.

Exercises 5.8 **67.** 12. Your answer will read "12;0," but of course the ";0" here is unnecessary, though it makes no difference. **68.** 13;2. Again, your answer will read "13;20," but the "0" at the end is unnecessary, though it makes no difference. **69.** 12465;702. **6↊.** 3552;0085.

Exercises 5.9 **6↋.** **6↋(a)** Negative. **6↋(b)** Positive. **6↋(c)** Positive. **6↋(d)** Negative. **6↋(e)** Negative. **6↋(f)** Positive. **6↋(g)** Positive.

Exercises 5.↊ **70.** **70(a)** 13 ÷ 2 **70(b)** 4 ÷ 3 **70(c)** $8 \div \frac{1}{3}$ **70(d)** $\frac{3}{4} \div \frac{2}{3}$ **70(e)** 356 ÷ 8 **70(f)** 18 ÷ 24 Note that there may be more groups than total items, in which case we will be dealing with fractions.

Exercises 5.↋ **71.** 9 **72.** 3 **73.** ↊ **74.** 8 **75.** 6 **76.** 9 **77.** ↋ **78.** 7 **79.** ↋ **7↊.** ↊ **7↋.** 3 **80.** 8

Exercises 5.10 **81.** Undefined. **82.** 8;73. **83.** 0;0004839. **84.** 4;8. **85.** 0;00000↊. **86.** 0;↋643. **87.** 1.

Exercises 5.11 **88.** **88(a)** 4. 5 × 6 = 26, and 5 × 7 = 2↋, so we know that the quotient is 6 (that is, we can't fit any more groups of size 5 into 2↊than 6). Then, we count how many are left over; or take the *difference*. 2↊ − 26 = 4. **88(b)** 1. ↊ × ↊ = 84, while ↊ × ↋ = 92. Since 92 is higher than 85, we know the quotient is ↊. Then, take the *difference*: 85 − 84 = 1. **88(c)** 3. 7 × 9 = 53, 7 × ↊ = 5↊. 5↊ is greater than 56, so our quotient is 9. Then take the difference: 57 − 53 = 4. **88(d)** 1. 9 × ↋ = 83, 9 × 10 = 90. 84 − 83 = 1. **88(e)** 2. 6 × 8 = 40, and 6 × 9 = 46. So our quotient is 8. 42 − 40 = 2. **88(f)** 0. ↊ × ↊ = 84; 84 − 84 = 0.

Exercises 5.12 **89.** **89(a)** 2, 3, 4, 6. (We can omit 1 and 10, because 1 and the number itself are factors of all numbers; in other words, we already know them.) **89(b)** 2, 4. **89(c)** 2, 7. **89(d)** 2, 4, 8. **89(e)** None (other than 1 and itself); it is prime. **89(f)** 2, 3, 6, 7. **89(g)** 2. **8Ƨ.** **8Ƨ(a)** 2, 2, 3. **8Ƨ(b)** 2, 2. **8Ƨ(c)** 2, 7. **8Ƨ(d)** 2, 2, 2, 2. **8Ƨ(e)** 11. **8Ƨ(f)** 2, 3, 7. **8Ƨ(g)** 2. **8Ɛ.** **8Ɛ(a)** 2, 3, 6, or 9 people. Note that you don't need to actually divide 46 by every number between 2 and 10; use the divisibility tests. **8Ɛ(b)** 2, 4. (Also 5 or Ƨ, but we don't have easy tests for that.) **8Ɛ(c)** 2, 4, 8. (Also 5 or Ƨ, but again we don't have easy tests for that.) **8Ɛ(d)** 2, 3, 4, 6, 8, 9, 10.

Exercises 5.13 **90.** **90(a)** 10Ɛ, R0. **90(b)** 352, R2. **90(c)** 4875, R0. **90(d)** 1503, R5. **90(e)** 593, R2. **90(f)** 15Ƨ26Ɛ, R2. **91.** **91(a)** The question is 185 ÷ 3 (3 because yourself and your two friends); so each of you must pay $69. **91(b)** 185 ÷ 5; so each of you must pay $41. (Though we're ignoring fractions and remainders, this one happens to be exact.) **91(c)** 185 ÷ 9; so each of you must pay $23. **92.** **92(a)** 355 ÷ 7 = 5Ɛ **92(b)** 86 ÷ 12 = 7, with a remainder. So get 8; otherwise you'll be short. **92(c)** 86 ÷ 12 = 7, R4. So you'll be 4 short. **92(d)** You can answer this by multiplication and subtraction: you've ordered 8 plattes of 12 crates each, so 8 × 12 = 94; then find the difference between 94 and the 86 crates you need, which is Ƨ. Or, since you've already done 86 ÷ 12 = 7, R4, you can simply remember that each platte is 12 creates, and that one more platte means you'll have 12 more. 12 additional crates minus the four you were short; 12 − 4 = Ƨ. **93.** **93(a)** $1;20. **93(b)** You've already figured that each pound is $1;20; so 1;2 × 7 = 8;2, or $8;20.

Exercises 5.14 **94(a)** 32, R10. **94(b)** 42, R7Ɛ. **94(c)** 200, R38. **94(d)** 174, R453. **94(e)** Ɛ, R4240. **94(f)** ƧƧ1, R35. **95.** **95(a)** $741. (Remember that we are ignoring remainder and fractions.) **95(b)** $373. **95(c)** $598. **95(d)** $Ƨ716. **96.** **96(a)** 827 × 0;9 = 61Ɛ;3, so $61Ɛ;30. This is a multiplication question; just as learning how much 3 tons cost would be 827 × 3, learning how much 0;9 costs is 827 × 0;9. **96(b)** Break this problem up. If 0;9 ton was $6Ƨ2;30, how much was 0;1 ton? 6Ƨ2;3 ÷ 9 = 91;7, or $91;70 for 0;1 ton. Now multiply by 10 (since 1 is 0;1 × 10): $917;00, which we can do just by moving the dit. Check the answer: 917;00 × 0;9 = 6Ƨ2;3, our original price. So hay was $917;00 per ton last year. **96(c)** Again, find the cost of 0;1 ton. 0;1 ton is 0;9 ton divided by 9; 827 ÷ 9 = ƧƐ;54, or $ƧƐ;54 for 0;1 ton. You need 0;7 ton, which is 7 times 0;1 ton. ƧƐ;54 × 7 = 648;14, or $648;14 for 0;7 ton.

Exercises 5.15 **97.** **97(a)** 54;9. **97(b)** 885;7249. **97(c)** 17;471Ɛ. **97(d)** 20;0696. **97(e)** 2;0703. **97(f)** 0;ƐƧ25. **97(g)** 0;7Ƨ36. If you're having trouble with this problem, remember that, when you shift the divisor's dit to the right 3 places, you'll also need to shift the dividend's dit to the right 3 places; so you'll have to add a 0 to the dividend's right in this case. This also gives you a divisor larger than your dividend; you'll have to add another zero to make this work. The problem you'll end up actually solving will be Ƨ7Ƨ0;0 ÷ 14327. **98.** You need to do two problems for this; first, find the number of reams, and then find the price per ream. 12 reams in each of 44 boxes is 12 × 44 = 508 reams; then do Ƨ6;Ƨ2 ÷ 508 = 0;2111, or $0;21 per ream. **99.** Again, you must do two problems here: how much has your stock of rabbits increased, and then how many has it increased per day? The first question is a question of *difference*; 1Ƨ1 − 56 = 147. Then, divide the difference by the number of days over which the increase occurred. That was four weeks; four weeks times seven days per week is $4 \times 7 = 24$. So $147 \div 24 = 7;1351$ new rabbits per day. **9↊.** Presumably we're looking for an average here; so we add the grades, $↊4 + ↋4 = 198$. Divide that by the number of tests: $198 \div 2 = ↊↊$.

EXERCISES 5.16 **9↋.** **9↋(a)** Not divisible by 7. **9↋(b)** Divisible by 7. **9↋(c)** Divisible by 7. **9↋(d)** Not divisible by 7. **↊0.** **↊0(a)** 1↊15. **↊0(b)** 1409;3518; R2. **↊0(c)** 21481;4972; R3.

EXERCISES 5.17 **↊1.** **↊1(a)** 1, 2, 37, 72. **↊1(b)** 1, 67. (This, of course, is a prime.) **↊1(c)** 1, 2, 4, 17, 32, 64. **↊1(d)** 1, 2, 4, 17↋, 33↊, 678. **↊1(e)** 1, 2, 5, ↊, 11↊7, 2392, 594↋, ↋69↊. **↊1(f)** 1, 3, 7, 9, ↋, 19, 29, 53, 65, 83, 173, 499. **↊1(g)** 1, 2, 11, 22. **↊1(h)** 1, 3, 7, 19, 1E, 59, 115, 343.

EXERCISES 5.18 **↊2.** **↊2(a)** Numerator: 4; denominator, ↊. **↊2(b)** Numerator: 8; denominator, 9. **↊2(c)** Numerator: ↊; denominator, 4. **↊2(d)** Numerator: 3; denominator, 2. **↊2(e)** Numerator: 134; denominator, 87. **↊3.** **↊3(a)** Proper. **↊3(b)** Proper. **↊3(c)** Improper mixed. **↊3(d)** Proper mixed. **↊3(e)** Improper.

EXERCISES 5.19 **↊4.** **↊4(a)** 6. **↊4(b)** 5. **↊4(c)** 1. **↊4(d)** 4. **↊4(e)** 3. **↊4(f)** 34. **↊4(g)** 93.

EXERCISES 5.1↊ **↊5.** **↊5(a)** 102↋9. **↊5(b)** 518↊. **↊5(c)** 20. **↊5(d)** 320. **↊5(e)** 36. **↊5(f)** 96. **↊5(g)** 430↋ 5866.

EXERCISES 5.1↋ **↊6.** **↊6(a)** 3. Really, $3\frac{0}{6}$, but $\frac{0}{6}$ is 0, so there's no need to write it. **↊6(b)** $7\frac{2}{4}$. **↊6(c)** $↊\frac{3}{↊}$. **↊6(d)** $5\frac{1}{3}$. **↊6(e)** $5\frac{1}{4}$.

EXERCISES 5.20 **↊7.** **↊7(a)** $\frac{5↋}{9}$. **↊7(b)** $\frac{17}{4}$. **↊7(c)** $\frac{136}{14}$. **↊7(d)** $\frac{5}{2}$. **↊7(e)** $\frac{↋}{3}$.

EXERCISES 5.21 **↊8.** **↊8(a)** $\frac{1}{49}$. **↊8(b)** $\frac{8}{11}$. **↊8(c)** $\frac{9}{12}$. **↊8(d)** $\frac{1}{2}$. **↊8(e)** $\frac{9}{24}$. **↊8(f)** $\frac{1}{5}$. **↊8(g)** $\frac{2}{9}$. **↊8(h)** $\frac{5}{↋}$. **↊8(i)** $\frac{15}{↊↋}$. **↊8(j)** $\frac{1}{3}$.

EXERCISES 5.22 **↊9.** $\frac{47}{53}$. **↊↊.** $\frac{59}{68}$. **↊↋.** $\frac{15}{10}$, or $1\frac{5}{10}$. **↋0.** $\frac{1}{2}$. More precisely, $\frac{4}{8}$; but remember to *reduce* the fraction! **↋1.** $\frac{119}{214}$. **↋2.** $\frac{1}{2}$. Again, more precisely, $\frac{↋}{1↊}$; but remember to *reduce* the fraction! **↋3.** 1. **↋4.** $\frac{2}{8}$. **↋5.** A tricky one, as it involves negatives. Subtract as normal, following the same rules for negatives in addition and subtraction as always. $-\frac{3↋5}{1280}$. **↋6.** $\frac{24}{47}$. **↋7.** $\frac{1}{2}$. **↋8.** 0. **↋9.** The two pizzas are each divided into eight slices, making 14 total slices of pizza. Since we need to find the part of the total, we make fractions based on the entire amount. So $\frac{3}{14} + \frac{5}{14} = \frac{8}{14}$, or $\frac{1}{2}$ of the whole amount of pizza. **↋↊.** 14 $(8 + 8)$ slices of pizza; $\frac{14}{14} - \frac{8}{14} = \frac{8}{14} = \frac{1}{2}$ of the total amount of pizza is available. **↋↋.** $340 - 216 = 126$ reams were sold (since 216 is how many you have left). So you have shipped $\frac{126}{340} = \frac{25}{68}$ of your stock.

EXERCISES 5.23 **100.** $\frac{28}{53}$. **101.** $\frac{1}{4}$. **102.** $\frac{1}{2}$. **103.** $\frac{1}{2}$. **104.** $\frac{29}{34}$. **105.** $\frac{2}{7}$. **106.** $\frac{10}{↋1}$. **107.** $\frac{32}{19}$. **108.** **108(a)** Multiply: $\frac{1}{3} \times \frac{1}{4} = \frac{1}{10}$. **108(b)** Tougher. The easiest way is to make 0;9 into a vulgar fraction, which is easy enough: $\frac{9}{10}$. Then multiply: $\frac{9}{10} \times \frac{1}{3} = \frac{9}{30}$. Reduce that answer to $\frac{1}{4}$. Alternatively, turn $\frac{1}{3}$ into a digital fraction, then multiply: $0;9 \times 0;4 = 0;3$, which is $\frac{1}{4}$. **109.** **109(a)** $5\frac{1}{20}$. $34\frac{1}{3} = \frac{↊1}{3}$. $\frac{↊1}{3} \div 8 = \frac{↊1}{3} \div \frac{8}{1} = \frac{↊1}{3} \times \frac{1}{8} = \frac{↊1}{20} = 5\frac{1}{20}$. **109(b)** $↊\frac{1}{10}$. $34\frac{1}{3} = \frac{↊1}{3}$. $\frac{↊1}{3} \div 4 = \frac{↊1}{3} \div \frac{4}{1} = \frac{↊1}{3} \times \frac{1}{4} = \frac{↊1}{10} = ↊\frac{1}{10}$. **109(c)** $1\frac{5}{6}$. $34\frac{1}{3} = \frac{↊1}{3}$. $\frac{↊1}{3} \div 1↊ = \frac{↊1}{3} \div \frac{1↊}{1} = \frac{↊1}{3} \times \frac{1}{1↊} = \frac{↊1}{56} = 1\frac{47}{56} = 1\frac{5}{6}$. **10↊.** You had 48 boxes of staples, and have shipped $\frac{3}{4}$ of them. $(48 \times \frac{3}{4}) = (\frac{48}{1} \times \frac{3}{4}) = \frac{120}{4} = 36$ boxes were shipped. $48 - 36 = 12$ boxes are left. **10↋.** $\$46 \times 134 = \$5900$, and $\frac{5900}{6500} = \frac{59}{65}$; this is the portion of your budget spent on meals. This leaves $(1 - \frac{59}{65}) = (\frac{1}{1} - \frac{59}{65}) = (\frac{65}{65} - \frac{59}{65}) = \frac{8}{65}$ for your cake.

EXERCISES 5.24 **110.** 4 **111.** 3 **112.** 17 **113.** 20 **114.** 8 **115.** 2 **116.** 18

117.6

Exercises from Chapter 6

Exercises 6.1 **1.** **1(a)** 5;4389 **1(b)** 834;3992 **1(c)** 93;5782 **2.** **2(a)** 5;4389 **2(b)** 834;3993 **2(c)** 93;5783 **3.** **3(a)** 59000 **3(b)** 5Ƨ000 **3(c)** Ɛ6000 **3(d)** Ƨ0000. Note that 6 means we round *up* the preceding digit; but the preceding digit is Ɛ, so we must round that up to 10; so we must add the 1 to the digit preceding *that*, the make the triqua digit 0.

Exercises 6.2 **4.** **4(a)** Two. **4(b)** Four. **4(c)** Four; leading zeroes are not significant, while trailing zeroes when there is a radix point are significant. **4(d)** At least five. "33001" are all clearly significant; the final "0" may or may not be, depending on information we do not have. **5.** No; your answer has three significant digits, while your measured quantities both only have two. You are limited by your least accurate number; so you round to two significant digits, or Ɛ;0. (Ƨ;Ɛ9; you round that up, but that rolls over the Ɛ, as well, so you need to round the Ƨ up to Ɛ and the Ɛ back down to 0.) **6.** No. 0;65 Maz to one significant digit must be rounded down, to 0;6. You should *not* rely on this bridge holding up your train. Strengthen it.

Exercises 6.3 **7.** 14 **8.** 1323 **9.** 368 **Ƨ.** 8 **Ɛ.** 194 **10.** 69 0000 **11.** 1481 **12.** 31Ɛ14 **13.** **13(a)** 21 **13(b)** 681 **13(c)** 974Ɛ61 **13(d)** 5684 **14.** **14(a)** 368 **14(b)** 851768 **14(c)** 8 **14(d)** 4768 **15.** 1309054

Exercises 6.4 **16.** 714 **17.** 3969 **18.** 346 0000 **19.** 6. Remember that any number to the the power of 1 is itself. **1Ƨ.** 368 **1Ɛ.** 714 **20.** 69919101 **21.** 16Ɛ68 **22.** 1000 **23.** 9061 **24.** 257854 **25.** 14 **26.** 69

Exercises 6.5 **27.** The same as $\frac{1}{8^1}$, and raising a base to an exponent of one equals itself; so this is the same as $\frac{1}{8}$, or 0;16. **28.** $\frac{1}{8^3}$, 0;00346. **29.** 0;0001. **2Ƨ.** 0;00000194. **2Ɛ.** 8. Remember to simply add the exponents; $-7+8=1$, and $8^1=8$. **30.** 1323. Remember to simply subtract the exponents; $3--4=3+4=7$, and $3^7=1323$. It's quite easy to subtract the exponents and come up with -1, and therefore think the answer is 3^{-1}, or 0;4. Remember the rules for negative numbers! **31.** 0;00000Ɛ483. Again, remember the rules for negative numbers; here, when we multiply our exponents we multiply a negative by a positive, giving a negative result. So our number is 8^{-6}, which is quite small. **32.** 107854. Here, we are multiplying two negatives, which yields a positive; so our problem is really 4^9 $(-3\times-3=9)$, or 107854.

Exercises 6.6 **33.** 90 **34.** 140 **35.** 1000 **36.** 232 **37.** 238 **38.** -360 **39.** -708

Exercises 6.7 **3Ƨ.** **3Ƨ(a)** $12000(1+0;03)^6 =$ 13Ƨ15;Ɛ4 **3Ƨ(b)** Take the new account value and subtract the initial deposit. 13Ƨ15;Ɛ4 − 12000 = 1Ƨ15;Ɛ4 is the money the investor has gained.

Exercises 6.8 **3Ɛ.** Radicand : power; degree : exponent; root : base. **40.** Its *cube* root, because the power of a number raised to an exponent of 3 is called its *cube*. **41.** $2\cdot2\cdot2=8$

Exercises 6.9 **42.** **42(a)** $9^{\frac{1}{3}}$ Remember that, if the radicand has no exponent, use 1. **42(b)** $\sqrt[7]{5^6}$ **42(c)** $\sqrt[1]{4^4}$; the first root of a radicand is itself. **42(d)** $8^{\frac{3}{2}}$

Exercises 6.Ƨ **43.** 5;475 **44.** 3;Ɛ0ƐƧ **45.** 21Ƨ;4217

Exercises 6.↋ **46.** **46(a)** 1;X490 + 2;↊975 = 4;9245. The antilog of 4;9245 is 6871↊;1179; the true answer to four digits is 68740;9852. Notice that truncating our logs throws our answer quite far off; eight-digit logs would have given us an answer with four digits of accuracy. **46(b)** 0;6↋↊2 + 0;65↊5 = 1;1586. The antilog of 1;1586 is 14;3573; the true answer is 14;36, with no further places. **46(c)** 0;5849; in this case, the true answer to four digits is the same. **46(d)** 1784;675↋. Again, this is the same as the true answer. **46(e)** 50;6741. Again, this is the same as the true answer. **46(f)** Easily done mentally with your multiplication table; but offered to show that the logarithmic method works. You will likely get an answer very close to 4, if not exactly. The author's calculator gave 3;↋↋↋↋.

Exercises 6.10 **47.** **47(a)** 130 : 1, or 130ft : 1s. This is a rate. **47(b)** 1 : 2 : 4. This is *not* a rate, because even though each part is a different substance, the *units* ("parts") are the same for each. **47(c)** 2 : 1 hydrogen to oxygen, or 1 : 2 oxygen to hydrogen. This is *not* a rate, because the unit is the same in both cases (namely, "atoms").

Exercises 6.11 **48.** If we set Jim's age equal to 1, then we know that 1;6 (one and a half) of Jim's age is 30. Thus, we have two ratios we can set equal to one another: 1;6 : 1 (Jim's age plus half of itself to Jim's age) and 30 : x (30, which is Jim's age plus half of itself, to the unknown quantity, Jim's actual age). $\frac{1;6}{1} = \frac{30}{x}$. Cross-multiply; 1;6$x$ = 30; $\frac{30}{1;6} = x$; $x = 20$. So Jim is, in fact, 20. Let's check it; half of 20 is 10, and 20 + 10 = 30. We got it! **49.** Take two ratios: $\frac{3}{3;6} = \frac{4}{x}$. Cross-multiply; four pounds is \$4;80. Quicker and easier than the other method we have used, determining the price of 1 pound, then multiplying by 4. **4↊.** 76 is this number, plus $\frac{3}{7}$ of this number. So 76 is $1 + \frac{3}{7}$, or $\frac{↊}{7}$. $\frac{↊}{7}$ is then the numerator over 1, our number; ten the other ratio is 76 (the $\frac{↊}{7}$) over x (the unknown number). So $\frac{↊ \div 7}{1} = \frac{76}{x}$. Cross-multiply; $\frac{↊}{7}x = 76$; this is the same as $\frac{↊x}{7} = 76$. Multiply both sides by 7 to get rid of the denominator on the left; ↊x = 446; $x = 53$. So 53, increased by $\frac{3}{7}$ of 53, equals 76. Let's check this, though. $\frac{3}{7}$ of 53 is 23. 53 + 23 = 76. We did it! **4↋.** The first receives \$1484;6↊34, the second receives \$358↋5184. (With our answers truncated to four places.) **50.** \$3 5324. (Your partner gets \$6 ↊648.) **51.** 14 feet and 8 feet. **52.** $\frac{3}{5}$ (more specifically, $\frac{16}{26}$; but remember to reduce the fraction).

Exercises 6.12 **53.** **53(a)** The distance travelled and the time taken to travel it are directly proportional, and the speed is the proportionality constant. If he travels longer, the distance he has travelled increases; if the distance he has travelled has increased, it must have taken him longer to travel it. The constant ratio by which he must multiply either his distance or his time to determine the other is his speed. **53(b)** The force pulling the drill is directly proportional to the mass of the drill. To determine the force, one would multiply the mass by the acceleration of gravity; the latter is the proportionality constant. **53(c)** These are all important things to think about, but none of them are directly proportional to any other. **53(d)** The circumference of a circle is directly proportional to its diameter; π is the proportionality constant. **54.** **54(a)** 114;279↋; 43 × 3;1848. **54(b)** ↊0;261↊; 32;32 × 3;1848. **54(c)** 32;↊00↊; ↊2 = 3;1848 × x, so divide 3;1848 from both sides; ↊2 ÷ 3;1848. **54(d)** 1;↊137.

Exercises 6.13 **55.** **55(a)** This *is* inverse proportion; the number of hours and the number of painters are the values, and anytime you increase one the other decreases. (More painters, less time; more time, less painters.) **55(b)** The proportionality constant is the product of the two terms. Two people and eight hours makes 14. **55(c)** Two people can paint it in eight hours, and $2 \times 8 = 14$. We need to complete it in four hours; so replace the 8 with a 4. What times 4 equals 14? Or $x \times 4 = 14$? To answer that, we use division: $x = \frac{14}{4} = 4$. We need at least four people painting to finish in four hours. **56.** **56(a)** $y = \frac{1}{x}k$; we need to learn the proportionality constant. Fortunately, we know that $y = \frac{1}{x}k$ is the same as $yx = k$, and we know that at 2;8 au, the gravitational attraction is 1. We need to use the *square* of the distance, though. So $1 \times 2;8^2 = 7;14$; 7;14 is the proportionality constant. Now we plug in our new distance, 2 au: $y = \left(\frac{1}{2^2}\right) 7;14$; solve for y, and we get 1;94. **56(b)** Same process. The proportionality constant is $yx^2 = 8Ɛ \times 234^2 = 3Ƨ3\,18Ƨ8$. Our new distance is 2Ƨ2, so $y = \frac{1}{2Ƨ2^2} 3Ƨ3\,18Ƨ8$. Solve for y; the answer is 58;5915 (rounded to four digits).

Exercises 6.14 **57.** **57(a)** Ƨ0%. **57(b)** 94%. **57(c)** 37%. **57(d)** 80%. **58.** **58(a)** 47% **58(b)** 0;6321; don't be fooled by the perbiqua itself being a digital fraction. Move the dit, as always. **58(c)** 143;9% **58(d)** 0;3% **59.** Ƨ0%; 20 students passed. **5Ƨ.** Cross-multiply; $\frac{x}{340} = \frac{Ɛ0}{100}$. 308 students passed.

Exercises 6.15 **5Ɛ.** Mean: 93;16; median: Ƨ5; mode: Ƨ5 and Ƨ8 (or none); range: 7Ɛ. **60.** Mean: 85;3518; median: 86; mode: none; range: 52.

Appendix B

Resources for Further Study

The following is arranged roughly by topic, as indicated by headings placed appropriately in the text. Within each topic, rather than ordering alphabetically, the ordering has been (somewhat arbitrarily) made to follow simplicity and importance in the dozenal community. So those closer to the beginning of a topic are, all other things being equal, simpler, or more important in the dozenal community, or both.

Basic Dozenals

Malone, James. *Eggsactly a Dozen*. http://www.dozenal.org/drupal/content/eggsactly-dozen. Likely the best simply introduction to non-decimal (specifically, dozenal) counting in existence. It explains the nature of counting by dozens clearly, quickly, and simply. A great introductory text.

Dozenal Society of America. *A Dozenal Primer*. http://www.dozenal.org/drupal/content/a_dozenal_primer.html. A quick, dozen-page review of the dozenal system, why it's preferable to decimal, and what can be done with it.

Zirkel, Gene. *A Brief Introduction to Dozenal Counting*. http://www.dozenal.org/drupal/content/brief-introduction-dozenal-counting. A slightly more in-depth presentation of dozenals, including a multiplication table; this article briefly explains how we wound up counting in decimal and why we should think better of it.

Schiffman, Jay. *Fundamental Operations in the Duodecimal System*. http://www.dozenal.org/drupal/content/fundamental-operations-duodecimal-system. Gives in some detail the algorithms used for the four functions of basic arithmetic, with a specific regard for dozenal. Also explains how to convert from dozenal to decimal and back again. Obviously, not as detailed as this text; but quite a good review text for someone already comfortable with these algorithms but needing a refresher.

Zirkel, Gene. *Decimal-Dozenal Conversion Rules*. http://www.dozenal.org/drupal/content/decimal-dozenal-conversion-rules. Explains and demonstrates how to convert dozenal numbers to decimal and back again by a number of different methods, according to nine simple rules.

DeVlieger, Michael. *Featured Figures: Basic Operations*. http://www.dozenal.org/drupal/content/featured-figures-basic-operations. A simple, one-page document showing the dozenal addition and multiplication tables.

Treisaran. *Dozenal Divisibility Tests Quick Guide*. http://www.dozenal.org/drupal/content/dozenal-divisibility-tests-quick-guide. A great guide to divisibility tests in dozenal, including some relatively complicated ones for five and seven, but also for the simpler tests that we've seen in this book.

Dozenal Society of America. *Manual of the Dozenal System*. http://www.dozenal.org/drupal/content/revised_manual_dozenal_system.html. A thorough examination of dozenal, including rules of conversion and arithmetic; essentially a more complete version of *A Dozenal Primer*, above. Likely the most complete, yet still short, introduction to dozenals.

None. *The Aspirant's Tests*. http://www.dozenal.org/drupal/content/aspirants-tests. Originally, the Dozenal Society of America required "aspirants" to pass four tests, showing their knowledge of dozenals. This document gives those tests, with answers.

Andrews, F. Emerson. New Numbers: How Acceptance of a Duodecimal (12) Base Would Simply Mathematics. Harcourt, Brace and Co., 1153. The follow-up to his groundbreaking article in The Atlantic Monthly, *An Excursion in Numbers*, which started the modern dozenal movement, Andrews gives a book-length argument for dozenals and explains why they are so useful. Out of print; still available used in many places.

———. *An Excursion in Numbers*. http://www.dozenal.org/drupal/content/excursion-numbers.html. The groundbreaking article, originally published in The Atlantic Monthly, which kicked off the modern dozenal movement. Still very much worth reading.

Essig, Jean. Douze, notre dix futur : Essai sur la numération duodécimale et un système métrique concordant. Dunod: Paris, 116Ɛ. The first modern non-English work on dozenals, this contains a defense and explanation of dozenals, along with a complete *Système Internationale* twelve-based metric system.

Metric Systems

Pendlebury, Tom and Donald Goodman. TGM: A Coherent Dozenal Metrology. http://www.dozenal.org/drupal/content/tgm-coherent-dozenal-metrology. A detailed exposition of the complete, coherent dozenal metric system known as TGM, after its three primary units (the Tim; the Grafut; and the Maz). Shows some of the immense advantages that can come from dozenals in the field of mensuration, and also presents a practical system for dozenal measurement.

Suga, Takashi. *Proposal for the Universal Unit System*. http://www.dozenal.com. A proposed system of measurement based upon fundamental constants. While it declaims any attempt to be practically useful, the system is an interesting showing of how dozens can direct our understanding of the universe.

Advanced Mathematics

DeVlieger, Michael. *Dozenal Frequently Asked Questions*. http://www.dozenal.org/drupal/content/dozenal-frequently-asked-questions. This article's name is a bit deceptive; for while these questions may be frequently asked, the article containing them provides a

very detailed, very precise, and very scholarly explanation for the superiority of dozenal arithmetic. Very suitable for mathematicians; perhaps out of the depth of only casually interested students. But worth attempting for anyone.

Terry, George Skelton. *Duodecimal Arithmetic*. Longmans, Green and Co., 1156. A groundbreaking book, Terry produced extension dozenal tables for logarithms and trigonometic functions, along with a basic explanation of dozenal arithmetic in its preface. Long out of print, the work can still be found in many places used. Of limited utility in the modern day, given digital calculators replacing the need for mathematical tables; but great for the interested collector.

History of Dozenal Arithmetic

Osburn, Christopher. *Some Notes on the History and Desirability of Using Alternate Number Bases in Arithmetic*. http://www.dozenal.org/drupal/content/some-notes-history-and-desirability-using-alternate-number-bases-arithmetic.html. A brief discussion of the major historical non-decimal numbers systems and some of the chief benefits of each. Divides bases sensibly, between "finger bases" (five, ten, twenty), "binary bases" (two, eight, sixteen); and "other" (sixty, twelve).

Zirkel, Gene. *A History of the DSA*. http://www.dozenal.org/drupal/content/history-dsa.html. A discussion of the history and development of the Dozenal Society of America. Prof. Zirkel has been there, in a more or less active capacity, for nearly all of it, and knew the great founders of the Society personally. Consequently, this is a very informative discussion.

Zirkel, Gene. *Reflections on the DSGB*. http://www.dozenal.org/drupal/content/reflections-dsgb.html. While Prof. Zirkel wasn't personally involved in the Dozenal Society of Great Britain, he was very familiar with the development of the DSGB, and knew a few of the key players in that development. Compiled from notes from Brian Bishop, Shaun Ferguson, and Robert Carnaghan, all major figures in the DSGB.

Glaser, Anton. History of Binary and Other Nondecimal Numeration. Tomash Publishers: 1971. A very detailed history of numeration, particularly of binary, but also of other non-decimal bases. Sensibly split up between pre-Leibnitz, Leibnitz, and post-Leibnitz, with a lot about more current uses. Very scholarly and informative, but still an easy and comprehensible read. An impressive work of scholarship.

Appendix C

Glossary

abundant a number for which the sum of its factors, excluding itself, is greater than the number itself. **71**

addend one of the operators in an addition operation, which will be used to produce the sum. **38**, 39, 3↋, 40, 43, 44, 46, 51, 54, 98, ↊7, ↋1, 106, 127, *see also* addition, sum & summation

addition (+) the total of two or more numbers taken together. 33–35, 37, 39, 41, 43–46, 49, 4↊, 50–55, 58–5↊, 61–63, 69, 86, 97, 98, 9↋, ↊2, ↊4–↊6, ↋↊, ↋↋, 109–111, 131–135, *see also* addend, sum & summation

addition table a table listing all single digits of a number system on both the top row and the leftmost column, with the remainder of the columns showing the results of the addition of the corresponding figures. 39, 46, 66, 131

algebra that branch of mathematics which deals with manipulating numbers symbolically, rather than directly, typically labelling unknown values with a letter of the alphabet. 119, 121, 124

algorithm a procedure for calculation, expressed as a finite list of steps for calculating a function. 27, 29, **34**, 3↋, 41, 42, 46, 49, 4↊, 53, 54, 58, 59, 62, 68, 6↊, 73, 89, 8↊, 90, 94, 98, 9↋, ↊0, ↊8, 131, 132, 134, 135, 13↊, 13↋, *see also* function

antecedent the first term in a ratio. **117**, 118, 119

anticommutativity a property of an operation meaning that reversing the order of operations changes the result of that operation such that the result of one is the negation of the other. **34**, 35, 43–45, 4↊, 50, 62, 64, 134, 146, *see also* commutativity

antilog the reverse log function; given the logarithm and the base, the antilog function gives the power. **10↋**

arithmetic the study of quantity, particularly when combined with other quantity in various ways; often used for the four basic functions. 33

associativity a property of an operation meaning that, in an expression containing two or more of the same operator, the order in which the operation is performed does not change the result, as long as the order of the operands does not change. **34**, 35, 39, 43, 54, 56, ↊4, 132, 146

average the quotient of the sum of a list of addends and the number of those addends; a synonym for *mean* (or, more accurately, for *arithmetic mean*. **106**, 127, *see also* mean

base the number being repeatedly multiplied by itself in an exponentiation operation; the b in b^x; also, the same function in a logarithm. **↋3**, ↋4, ↋6–↋8, ↋↊, 101–103, 109–10↋, 111, 112, *see also* exponentiation, power & exponent

bicentennial English word indicating a two-hundred-year anniversary. 22

borrow the means by which additional places are gained from higher digits in the subtraction algorithm. 46–49, 135

carry the means by which the excess of intermediate sums are resolved in the addition and multiplication algorithms. 3↋–41, 46, 58, 59, 5↋–61, 133, 134

casting out elvs a method of checking arithmetic results by taking digit sums and casting out the elvs. ↊6

-cia suffix indicating a negative exponential increase in the SDN system. 17, 18

commutativity a property of an operation meaning that the order of the operands does not change the result. **34**, 35, 39, 41, 50, 54, 56–58, ↋↋, 132, 137, 146, *see also* anticommutativity

composite a natural number greater than 1 which is not prime. **71**, 89

consequent the second term in a ratio. **117**, 118, 119

cube the power of a number raised to an exponent of three; the y in $b^3 = y$. **↋5**, 108, 109

degree the exponent to which the root must be raised to equal the radicand; the z in $\sqrt[z]{x} = y$; if not explicitly noted, then 2. 102, **103**, 104, 105, 128

denominator the number of parts which make up the whole in a vulgar fraction; also (which is saying the same thing) the divisor in a division statement. **↊**, ↋, 10, 89, 91–9↋, ↊1, 104, 108, 119, 122, 126, 140

difference the distance between two operands; the result of a subtraction operation. **44**, 45, 46, 4↋, 6↊, 74, 75, ↊7, ↋1, ↋7, 10↊, 129, 135, *see also* subtraction, minuend & subtrahend

digit finger; place in a number; e.g., "the number 508 has three digits". 3, ↋, 10, 18, 1↊, 1↋, 27, 41, 42, 46–49, 55, 57–61, 6↋, 70, 73, 76, 78–7↊, 85, ↊6, ↊9–↋2, 100, 105, 107, 109, 10↋, 111–114, 123, 132–13↋, 142

digit sum the sum of the digits of a number. **↊6**, ↊7

digital fraction a fraction written inline in the form of descending powers of the number base, typically to the right of some type of radix point; e.g., ";". 9, **↋**, 10, 24, 61, 89, 93, 94, ↊9, ↊↊, 101, 104, 105, 117, 118, 126, 127, *see also* fraction, vulgar fraction, irrational fraction, nonterminating fraction, repeating fraction & nonrepeating fraction

direct proportion a proportion in which one variable is always the product of the other and a constant; y and x are directly proportional if $\frac{y}{x}$ is constant. 120, **121**, 122–124

dit (;) the radix point for the dozenal base, as pronounced vocally. 10, 17–19, 24, 41, 42, 49, 57, 58, 61, 68, 76, 82, 83, 85, 93, ↊↊–↋0, ↋2, 110, 127, 137, 141

dividend the whole from which the divisor will be taken in a division operation. **64**, 65, 66, 68–6↋, 72–80, 82–85, 8↊, 8↋, 92, 94–96, ↊0, ↊7, ↋2, 10↊, 13↋–142, *see also* division, divisor, quotient, remainder & modulation

division (÷, /, —) the number of times that one number is contained in another; inverse of multiplication. 33, 37, 62–6↋, 72, 73, 76, 79, 7↊, 82, 83, 86, 92–97, ↊0–↊2, ↊5, ↊7, ↋2, ↋3, ↋↊, 108, 10↊–110, 117, 122, 131, 13↋–142, *see also* dividend, divisor, quotient, remainder, modulation & integer division

divisor the part which will be taken from the whole in a division operation. **64**, 65, 66, 68–6↋, 72–75, 77, 78, 7↊–85, 8↊, 8↋, 92, 94–96, ↊0, ↊1, ↋2, 10↊, 13↋–141, *see also* division, dividend, quotient, remainder & modulation

domain the set of all possible inputs for a function. 34, *see also* function & range

dotransbicentennial English word indicating a hundred-seventy-five year anniversary. 22

dotranscentennial English word indicating a seventy-five year anniversary. 22

dozen (10) meaning twelve; the base of the dozenal (duodecimal) system. 3, 4, Ɛ, 17

dublog a shortened name for logarithms to base 2. **114**, 115, 117

elementary result a sum of the addition table or a product of the multiplication table, along with its corresponding inverse (difference and quotient). **131**, 133, 135, 13Ɛ, 140

-ennial suffix indicating that the preceeding is meant to apply to a year; e.g., an anniversary; so that *biquennial* means "200th anniversary". 22–24

equality (=) the state of having identical values. 14, 15

equation an expression containing a statement of equality, typically by the "=" sign. **14**, 15, 56, 90, 99, Ɛ4, Ɛ7, 100, 101, 108, 109, 119, 122, 124, 125

estimation the act of approximating a number's value. Ƨ9, Ɛ0

Euler's number (e) a number defined as the limit of $(1+\frac{1}{n})^n$ as n approaches infinity; the base of the natural logarithm; ≈ 2;87523606 . . . . **111**

even any number divisible by 2. 6Ɛ, **70**, 71, 87, 103, 104, 129, 137, 138

exponent the number of times the base should be multiplied by itself (minus 1) in an exponentiation problem; the x in b^x.. 1Ɛ, **Ɛ3**, Ɛ4–ƐƧ, 100–105, 108, 109, 10Ɛ, 111, 113, *see also* exponentiation, power & base

exponential growth describes a function in which the increase in that function's value is proportional to the function's current value; a prime example is compound interest. **102**

exponential notation notation expressing orders of magnitude by giving an abbreviated number, usually with a single integral digit, coupled with a multiplication factor of 10 to a certain power. 1Ɛ, 20, 22, *see also* scientific notation

exponentiation an operation describing multiplication of a base by itself a certain number of times, with the number of times superscripted after the base. 20, Ƨ6, Ƨ9, **Ɛ3**, Ɛ4, Ɛ6, Ɛ8, ƐƧ–105, 109–10Ɛ, 111, 113, *see also* exponent, power & base

factor any of multiple operands in a multiplication operations; or, a divisor which exactly divides a dividend, without any remainder. **54**, 61, 62, 6Ƨ, **6Ɛ**, 71, 86–8Ƨ, Ɛ1, Ɛ2, 10Ƨ–110, 136–13Ɛ, *see also* multiplication, multiplicand, multiplier & product

factoring table a table assembled during the process of factoring. 86

figure digit or digits as expressed in a number; e.g., "the figures in the number 508 are 5, 0, and 8". 3, 5, 27–2Ɛ

four functions the four basic functions of arithmetic; namely, addition, substraction, multiplication, and division. 33, 35, 37, 64, 73, 79, 89, 97, Ƨ6, Ƨ9, 100, *see also* addition, subtraction, multiplication & division

fraction a number consisting of less than a single unit; or a number consisting of a number of whole units and a partial unit. **9**, Ƨ–11, 17, 19, 1Ɛ, 27, 41, 42, 49, 61, 69, 6Ƨ, 71, 79, 7Ƨ, 81–85, 89, 91, 92, 94–Ƨ1, 104, 105, 114, 122, 140, *see also* vulgar fraction, digital fraction, irrational fraction, nonterminating fraction, repeating fraction & nonrepeating fraction

function a relation between a set of inputs and a set of permissible outputs such that each input is related to exactly one output. **33**, 34, 35, 37, 43, 44, 68, 69, 109, 146, *see also* algorithm, domain & range

great-gross (1000) a dozen dozen dozen; decimally expressed as one thousand, seven hundred and twenty-eight; in SDN, triqua. 5, 10, 17

greatest common factor the highest number which is a factor of all of two or more numbers. 71, **8Ɛ**, 8Ɛ–91, 96

gross (100) a dozen dozen; decimally expressed as one hundred forty-four; in SDN, biqua. 4, 5, Ɛ, 10, 17

highly composite a positive integer with more divisors than any smaller positive integer. **71**

Hindu-Arabic the set of digits which are typically used in place notation systems of writing numerals; developed in India and were transported to Europe via the Arab world. 27, 29, 2Ɛ

identity element that number for a given operation which, when performed, will not change the other operand's value; e.g., the identity element for addition is 0, because $x + 0 = x$, not changing the number. **35**, 39, 44, 56, 67

improper fraction a vulgar fraction expressing a greater number of parts than are present in a single whole; a vulgar fraction with a numerator larger than its denominator. Ɛ, 91, 92, 104

inequality ($\neq$) the state of having different values; also, an expression containing a statement of inequality, including comparisons. 14, **15**

integer ($\mathbb{Z}$) the set of all non-fractional negative and positive numbers, plus zero; mathematically, $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. **9**, Ƨ–12, 14, 19, 42, 57, 69, 6Ƨ, 71, 83, 89, 90, Ɛ9, 105, *see also* whole number & natural number

integer division division in which only the integral part, or the integral part with a remainder, are stated, ignoring any fractional part. **6Ƨ**, 73, *see also* division, divisor, dividend, quotient, remainder & modulation

inverse proportion a proportion in which the product of two variables is always constant, no matter how either variable changes; y and x are inversely proportional if $x \cdot y$ is constant. 120, 124, 125

irrational fraction a digital fraction which does not have a terminating or repeating expression; a digital fraction which continues indefinitely; a digital fraction which cannot be expressed as a ratio of integers. 10, 11, *see also* nonterminating fraction, fraction, digital fraction, repeating fraction & nonrepeating fraction

irrational number ($\mathbb{P}$) any number which cannot be expressed as the ratio between two integers; only those numbers with digital representations that neither repeat nor terminate. **11**

least common multiple the lowest multiple of two or more numbers; the lowest number which is a multiple of all of a list of two or more numbers. **8Ɛ**, 90, 91, 98, 99, 9Ɛ

linear for our purposes, progressing steadily in a 1 : 1 ratio, rather than logarithmically. 112, 115

log . 109–10Ɛ, 111, 112, 114, *see also* logarithm

logarithm (log) the inverse operation to exponentiation; the logarithm of a number is the exponent to which the base must be raised to produce that number; the y in $\log_b(x) = y$. Ƨ9, Ɛ6, **109**, 10Ƨ–115, 117

long division the algorithm by which division not susceptible to mental solution is performed. 63, 67, 6Ƨ, 6Ɛ, 72, 73, 76, 77, 79–7Ɛ, 81, 84, 86, 94, 13↋–141

mantissa either the fractional part of a common logarithm, or the significand of a number written in exponential notation. 1↋

mean . **127**, 128–12↊, *see also* average

median the central value of a data set, when that data set is enumerated from smallest to largest (or vice versa); if there is no such number because there is an even number of data points, the median is the mean between the two central data points. **128**, 129, 12↊

minuend the first operator in a subtraction operation. **43**, 44–46, 48, 4↊–51, ↋1, *see also* subtraction, subtrahend & difference

minus sign (−) the mark which distinguishes negative from positive numbers; the operator which indicates subtraction. 8, 4↊, 4↋, *see also* negative number, negative sign & subtraction

mixed number an improper fraction expressed as a combination of a whole number and a proper fraction; e.g., "$1\frac{1}{4}$" as opposed to "$\frac{5}{4}$". ↋, 91–93, 104

mode the most common value in a data set; if all values appear the same number of times, there is no mode; there may be multiple modes, if multiple data points share the maximum number of appearances. **128**, 129, 12↊

modulation (%, mod) the arithmetical operation in which the remainder of a division problem is the result. 66, **69**, 6↊, 78, 142, *see also* division, dividend, divisor, remainder & quotient

multiple a number which can be obtained by multiplying a number by another number; e.g., 6 is a multiple of 2, because $2 \times 3 = 6$. 28, 2↊, 8↋, 126, 137, 139

multiplicand the first operator for a multiplication operation; the number which is to be scaled. **54**, 56, 58, 59, 60, ↊0, 137–13↊, *see also* multiplication, multiplier, product & factor

multiplication (×,·,*) the scaling of one number by another; equivalent to repeated addition. 12, 13, 20, 33, 34, 37, 51–54, 56–5↋, 61–63, 66–6↋, 73, 86, 8↋, 96–98, 9↋–↊2, ↊5, ↊7, ↋1–↋3, ↋↊, 100, 103, 104, 10↊–111, 122, 131, 132, 137, 138, 13↊, *see also* multiplicand, multiplier, product & factor

multiplication table a table listing all single digits of a number system on both the top row and the leftmost column, with the remainder of the columns showing the results of the multiplication of the corresponding figures. 56, 59, 66, 69, 6↊, 73, 7↊, 88, 131

multiplier the second operator for a multiplication operation; the number by which the multiplicand is to be scaled. **54**, 56–61, ↊0, 135–13↊, *see also* multiplication, multiplicand, product & factor

natural logarithm (ln) logarithms to base e, or Euler's number. **111**, 112, 114

natural number ($\mathbb{N}^*$) "counting numbers"; the whole numbers or integers beginning at one (1) and continuing to infinity; mathematically, $\{1, 2, 3, \ldots\}$. **8**, 9, 11, 71, *see also* whole number & integer

negative number the set of numbers which are less than zero (0). 8, 9, 14, 42, 45, 4↊, 4↋, 62, 68, 90, ↋9, ↋↊, 103, 104, 109, *see also* negative sign & minus sign

negative sign (−) the mark which distinguishes negative from positive numbers. 8, *see also* negative number & minus sign

nonrepeating fraction a digital fraction which does not have a terminating expression and which does repeat a regular pattern. 10, 11, *see also* irrational fraction, fraction, digital fraction, nonterminating fraction & repeating fraction

nonterminating fraction a digital fraction which does not have a terminating expression; a digital fraction which continues indefinitely. 10, 24, 79, 105, 111, *see also* irrational fraction, fraction, digital fraction, repeating fraction & nonrepeating fraction

number any type of number, whether natural, whole, integer, real, or otherwise; used for quantifying something; used to express quantity. 3, 17, 11Ɛ

numerator the number of parts of a whole which have actually been quantified in a vulgar fraction; also (which is saying the same thing) the dividend in a division statement. **Ɛ**, Ɛ, 10, 89, 91, 92, 94–99, 9Ɛ, Ɛ1, 104, 108, 119, 126, 140

odd any number not divisible by 2. 6Ɛ–71, 87, 104, 129, 138

operand a figure which is being used in an operation; e.g., the addends, the subtrahend, the minued, and so forth. 43, 44, 50, 72, 96, Ɛ6, *see also* operation

operation some arithmetical action which is being done on an operand or operands; e.g., addition, subtraction, and so forth. 51, 62, 69, 6Ɛ, 72, 78, 86, 94, 97, 9Ɛ, Ɛ1–Ɛ6, Ɛ3, Ɛ5, Ɛ8, Ɛ9, ƐƐ, 102, 114, 119, 131, 138, 13Ɛ, 140, *see also* operand

order of magnitude an approximate unit of size judged by powers of the number's base; so, e.g., a number approximately 10 times larger than another differs by one order of magnitude, 100 times larger differs by two orders of magnitude, and so forth. **1Ɛ**, 20, 21, 112, 113, 136

parenthesis ((/)) a symbol for grouping mathematical operations in an equation; operations contained in parentheses are always done first, and multiple operations within parentheses are themselves done according to the normal order of operations. **ƐƐ**, 100

parity the evenness or oddness of a number; so an even number has a parity of 0, an odd number a parity of 1. 70

partial dividend that portion of the dividend that is addressed in each step of long division. 73–76, 78, 79, 7Ɛ, 80, 84, 85, *see also* dividend, divisor, quotient, remainder & modulation

Pendlebury notation notation expressing orders of magnitude by prefixing positive orders as superscripts and negative orders as subscripts. 1Ɛ–22

Pendlebury, Tom prominent dozenalist of the Dozenal Society of Great Britain, who devised, among other things, Pendlebury notation and the TGM measurement system. 1Ɛ, *see also* Pendlebury notation

per SDN particle indicating fractions. 24, *see also* fraction, repeating fraction, nonterminating fraction & irrational fraction

perbiqua (%) any ratio expressed as parts per 100; "per" plus "biqua". 6Ɛ, **126**, 127

percentage (%) any ratio expressed as parts per 84, written as "100" in decimal; from "per" and "centum," the latter being Latin for "hundred". **126**

perfect square a number whose square root is a simple integer; that is, has no fractional parts. **105**, 106

pi (π) an irrational, nonrepeating number;

the ratio of the diameter of a circle to its circumference; also, the number of radians in a straight angle. 10

place notation a system of writing numbers in which each digit's value depends upon its place among the other digits. **27**, 29, 2↊, 41

positive number the set of numbers which are greater than zero (0). 8, 9, 42, 45, 4↊–50, 62, 68, 71, 90, 91, ↋7, ↋↊, 103

power the answer to an exponeniation problem; the y in $b^x = y$; also, that number from which a logarithm is taken, or the x in $\log_b(x) = y$. ↋3, **↋4**, ↋5, ↋6, ↋8–↋↊, 102, 103, 10↊, 10↋, 112, 114, *see also* exponentiation, base & exponent

prime a natural number greater than 1 which is divisible only by itself and 1. **70**, 71, 88, 89

product the result of a multiplication operation; the multiplicand scaled by the multiplier. **54**, 56, 58, 62, 68, 6↊, 88, 8↋, 9↋–↊1, ↋1, ↋6–↋8, 10↊–110, 120, 121, 123, 124, 128, 136, 137, 139, *see also* multiplication, multiplicand, multiplier & factor

proper fraction a vulgar fraction expressing a smaller number of parts than are present in a single whole; a vulgar fraction with a numerator smaller than its denominator. ↋

proportion (∝, , ::) the relationship of two quantities such that a change in one always induces a change in the other, when such changes are related by a constant multiplier. **11↋**, 120, 121, 126, 13↊

proportionality constant for direct proportion, the non-zero constant which links two directly proportional values, or that by which either value can be multiplied to yield the other; for inverse proportion, the product of the two variables themselves, which itself remains constant; also known as the "constant of proportionality" or the "constant of variation". **121**, 122, 123, **124**, 125

-qua suffix indicating a positive exponential increase in the SDN system. 17, 18

quadrancentennial English word indicating a twenty-five year anniversary. 22

quasquicentennial English word indicating a hundred-twenty-five year anniversary. 22

quotient the answer in a division operation; that is, the number of times the divisor occurs in the dividend. **64**, 65, 68–6↊, 72–75, 78–85, 87, 8↊, 91–93, 95, 96, 99, ↊0, 106, 10↊, 117, 127, 13↋, 142, *see also* division, dividend, divisor, remainder & modulation

radical an unresolved root in an equation; any expression containing a radicand and a $\surd$; synonym for *surd*. 102, 103, 105

radicand the number from which a root is extracted; the x in $\sqrt[z]{x} = y$. 102, **103**, 104

radix point (;) the separator between the integral and fractional parts of a real number; in dozenal, a semicolon, while in decimal, period. ↋, 10, 17–19, 1↋–21, 41, 49, 57, 61, 82, 83, 95, 96, ↊↋–↋2

range the difference between the largest and smallest data points in a set. **129**, 12↊

range the set of all possible outputs for a function. 34, *see also* function & domain

rate a special kind of ratio, which relates two quantities of different types to one another. **118**, 127

ratio (:) a mathematical expression of the relationship of one value to another.

Ɛ–11, 89, 115, **117**, 118–121, 126, 127

rational number ($\mathbb{Q}$) any number which can be expressed as the ratio between two integers; all integers, along with all fractions with a terminating or repeating digital representation, are rational. **11**

real number ($\mathbb{R}$) the entire spectrum of numbers from negative infinity to positive infinity, including those with fractional parts. **11**, 103

reciprocal a number which, when multiplied by another, yields a product of 1; the number which equals one divided by another number. **67**, 68, Ꮖ0, Ꮖ1, ƐᏖ, 125

reduction of fractions, the process of expressing a fractional value with the lowest possible numerator and denominator. **94**, 98, 140

remainder (R) when dividing, and when no fractional part is desired, the leftovers of the final group of the divisor which do not fit into the dividend; for this reason, by necessity it is lesser than the divisor. **66**, 69–6Ɛ, 74, 77–79, 81, 82, 84, 85, 8Ꮖ, 8Ɛ, 91–93, 142, *see also* division, dividend, divisor, modulation & quotient

repeating fraction a digital fraction which does not have a terminating expression, but which does repeat a regular pattern. 10, 11, 79, 89, *see also* irrational fraction, fraction, digital fraction, nonterminating fraction & nonrepeating fraction

root ($\sqrt{}$) that number which, when multiplied by itself a given number of times, will produce a certain number; the y in $y = \sqrt[z]{x}$. Ꮖ9, Ɛ3, 102, **103**, 104–10Ꮖ, 114, 127, 128

round to round is the act of approximating a number's value by selecting a place and adjusting it according to its following place, replacing all subsequent digits with zeroes. Ꮖ9, **ᏖᏖ**, ᏖƐ, Ɛ0, Ɛ2, 105, 114, 115, 12Ꮖ

scientific notation notation expressing orders of magnitude by giving an abbreviated number, usually with a single integral digit, coupled with a multiplication factor of 10 to a certain power. 1Ɛ, 20, 22, *see also* exponential notation

SDN (SDN) Systematic Dozenal Nomenclature. 22, *see also* Systematic Dozenal Nomenclature

sesquibicentennial English word indicating a two-hundred-fifty year anniversary. 22

sesquicentennial English word indicating a one-hundred-fifty year anniversary. 22

short division the algorithm by which division not susceptible to memorized tables but still amenable to mental solution is performed. 67, 84, 85, 13Ɛ

significand the significant digits of a number written in exponential notation; the numbers to the left of the "×" sign in such a number; also sometimes called the mantissa. 1Ɛ

significant digit those digits in a number which carry meaning according to the resolution currently available; often also called significant figures. Ꮖ9, **Ɛ0**, Ɛ1, Ɛ2

slide rule a mechanical calculator imprinted with logarithmic scales, enabling one to do multiplication and division, while a linear scale would only allow addition and subtraction. 110, 112

square the power of a number raised to an exponent of two; the y in $b^2 = y$. **Ɛ4**, Ɛ5, 103, 105–108, 10Ꮖ, 125, 127

subtraction ($-$) the difference between two numbers; the inverse of addition. 33, 37, 42–47, 4Ꮖ–52, 54, 61, 62, 66, 67,

69, 86, 97, 9Ɛ, 9Ɛ, Ɛ2, Ɛ4, Ɛ5, Ɛ7, ƐƐ, ƐƐ, 109–110, 131, 134, 135, *see also* minuend, subtrahend, difference & minus sign

subtrahend the second operator in a subtraction operation. **44**, 45, 46, 48, 4Ɛ–51, Ɛ1, *see also* subtraction, minuend & difference

sum ($\sum$) the result of an addition operation; the total of the addends. **38**, 39, 3Ɛ, 40, 44, 46, 51, 5Ɛ, 70, 71, Ɛ6, Ɛ7, Ɛ1, Ɛ6, 106, 10Ɛ, 10Ɛ, 127, 132, 134, 142, *see also* addend, addition & summation

summation ($\sum$) adding a sequence of numbers together; like addition, operates on addends and produces a sum. *see also* addition, addend & sum

surd an unresolved root in an equation; any expression containing a radicand and a $\sqrt{}$; synonym for *radical*. 102

Systematic Dozenal Nomenclature (SDN) a systematic and consistent system for referring to dozenal numbers. *see also* SDN

truncation the act of approximating a number's value by selecting a place and simply cutting off the remaining places, without other adjustment to the number. **Ɛ9**, ƐƐ, Ɛ0, 105, 12Ɛ

vulgar fraction a fraction written explicitly with the number of fractional parts along with the number of parts in a single whole. **9**, Ɛ–10, 13, 89, 92–98, Ɛ0, Ɛ1, Ɛ9, 104, 108, 117, 119, 126, *see also* fraction & digital fraction

whole number ($\mathbb{N}^0$) the natural numbers, with the addition of zero (0); the set of integers from zero to infinity; mathematically, $\{0, 1, 2, 3, \ldots\}$. **8**, Ɛ, 11, 6Ɛ, 91, 92, 94, Ɛ1, *see also* natural number & integer

zero (0) a symbol meaning nothing, no items; neither positive nor negative. **6**, 7, 8, 10, 17, 18, 27, 45, 57

Appendix D
Table of Definitions

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Appendix E

Appendix of Tables

Addition Table

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	2	3	4	5	6	7	8	9	↊	↋	10	11
2	3	4	5	6	7	8	9	↊	↋	10	11	12
3	4	5	6	7	8	9	↊	↋	10	11	12	13
4	5	6	7	8	9	↊	↋	10	11	12	13	14
5	6	7	8	9	↊	↋	10	11	12	13	14	15
6	7	8	9	↊	↋	10	11	12	13	14	15	16
7	8	9	↊	↋	10	11	12	13	14	15	16	17
8	9	↊	↋	10	11	12	13	14	15	16	17	18
9	↊	↋	10	11	12	13	14	15	16	17	18	19
↊	↋	10	11	12	13	14	15	16	17	18	19	1↊
↋	10	11	12	13	14	15	16	17	18	19	1↊	1↋
10	11	12	13	14	15	16	17	18	19	1↊	1↋	20

Multiplication Table

0	1	2	3	4	5	6	7	8	9	↊	↋	10
1	1	2	3	4	5	6	7	8	9	↊	↋	10
2	2	4	6	8	↊	10	12	14	16	18	1↊	20
3	3	6	9	10	13	16	19	20	23	26	29	30
4	4	8	10	14	18	20	24	28	30	34	38	40
5	5	↊	13	18	21	26	2↋	34	39	42	47	50
6	6	10	16	20	26	30	36	40	46	50	56	60
7	7	12	19	24	2↋	36	41	48	53	5↊	65	70
8	8	14	20	28	34	40	48	54	60	68	74	80
9	9	16	23	30	39	46	53	60	69	76	83	90
↊	↊	18	26	34	42	50	5↊	68	76	84	92	↊0
↋	↋	1↊	29	38	47	56	65	74	83	92	↊1	↋0
10	10	20	30	40	50	60	70	80	90	↊0	↋0	100

Divisor Tests	
Divisible by	If
2	Last is 0, 2, 4, 6, 8, or ↊
3	Last is 0, 3, 6, 9
4	Last is 0, 4, 8
6	Last is 0, 6
8	Second-to-last digit divides by 2 and last is 0 or 8
	Second-to-last digit doesn't divide by 2 and last is 4
9	Second-to-last digit is 0, 3, 6, or 9 and last is 0 or 9
	Second-to-last digit is 1, 4, 7, or ↊ and last is 6
	Second-to-last digit is 2, 5, 8, or ↋ and last is 3.
↋	Sum of digits is divisible by ↋; sum the digits of the sum, as many times as needed
10	Last is 0

Digital Expansions of Common Vulgar Fractions

Vulg. Frac.	Dig. Frac.	Vulg. Frac.	Dig. Frac.	Vulg. Frac.	Dig. Frac.
1/2	0;6	1/3	0;4	2/3	0;8
1/4	0;3	2/4	0;6	3/4	0;9
1/6	0;2	2/6	0;4	3/6	0;6
4/6	0;8	5/6	0;↊		
1/8	0;16	2/8	0;3	3/8	0;46
4/8	0;6	5/8	0;76	6/8	0;9
7/8	0;↊6				
1/14	0;09	2/14	0;16	3/14	0;23
4/14	0;3	5/14	0;39	6/14	0;46
7/14	0;53	8/14	0;6	9/14	0;69
↊/14	0;76	↋/14	0;83	10/14	0;9
11/14	0;99	12/14	0;↊6	13/14	0;↋3

Perfect Squares

$$
\begin{array}{rclcrcl}
1^2 & = & 1 & \quad & 2^2 & = & 4 \\
3^2 & = & 9 & & 4^2 & = & 14 \\
5^2 & = & 21 & & 6^2 & = & 30 \\
7^2 & = & 41 & & 8^2 & = & 54 \\
9^2 & = & 69 & & ↊^2 & = & 84 \\
↋^2 & = & ↊1 & & 10^2 & = & 100 \\
11^2 & = & 121 & & 12^2 & = & 144 \\
13^2 & = & 169 & & 14^2 & = & 194 \\
15^2 & = & 201 & & 16^2 & = & 230
\end{array}
$$

Colophon

This work was typeset entirely on the GNU/Linux operating system, using the `vi` *editor originally written by Bill Joy. It is coded and compiled using the LaTeX 2ε document preparation system, which is written atop the TeX typesetting system designed by Donald Knuth. Specifically, it was prepared with* `pdflatex` *from the TeXLive distribution, along with the* `pdfx`*,* `morewrites`*,* `xellipsis`*,* `tikz`*,* `gmp`*,* `microtype`*,* `xcolor`*,* `fontenc`*,* `lmodern`*,* `array`*,* `booktabs`*,* `tabularx`*,* `graphicx`*,* `amsmath`*,* `amsfonts`*,* `footmisc`*,* `amsthm`*,* `colortbl`*,* `tcolorbox`*,* `supertabular`*,* `longtable`*,* `lscape`*,* `textcomp`*,* `etoolbox`*,* `rotating`*,* `wrapfig`*,* `soul`*,* `lettrine`*,* `caption`*,* `calc`*,* `url`*,* `relsize`*,* `exscale`*,* `fancyvrb`*,* `parcolumns`*,* `paralist`*,* `dozenal`*,* `geometry`*,* `basicarith`*,* `verbatim`*,* `newfile`*,* `index`*,* `hyperref`*,* `footnotebackref`*,* `bookmark`*,* `glossaries`*, and* `glossary-mcols` *packages. It is set in Latin Modern 10 × 13. The digital document is standards-compliant PDF/A-2, Level B. GNU/Linux, TeX, LaTeX, and all the packages and materials contained herein are free, open-source, and freely available for use, distribution, and all other honorable applications.*